Dumas & Maenhout (2003) Central-Planning Approach: A Comment

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April 21, 2008

INCOMPLETE, NOT FOR QUOTATION

Abstract

The present note considers the central-planning approach in incomplete market put forward by Dumas & Maenhout (2003) and illustrates that it may not be used recursively with the proposed equilibrium automatic procedure. Indeed, the functional form postulated for the value function of the problem is not verified and, hence, the compensation mechanism fails to reproduce the compensated objective function. An illustration of the error induced by this approach is provided through the Basak & Cuoco (1998) limited participation example.

1 Introduction

The main attempt of the present note is to recast Dumas & Maenhout (2003) into a discrete time setting and proceed to show that the central-planning approach fails when used in a recursive fashion. The magnitude of the error induced by the latter procedure is illustrated through the Basak & Cuoco (1998) limited participation example.

Given the importance of the market incompleteness debate[1], it is of interest to have a general equilibrium computation methodology at one's disposal. In an attempt to achieve this goal, the dual approach in incomplete markets put forward by He & Pearson (1991) and Karatzas, Lehoczky, Shreve & Xu (1991) is fundamental. It allows not to postulate price processes *ab initio* and, hence, avoid the *tâtonement* approach.

Using the latter result when trading is discrete, a first equilibrium computation procedure is suggested by Cuoco & He (2001). They show that one is able to use a representative agent provided that one allows its Negishi weights to be stochastic. However, the central planner only serves to determine optimal consumptions and accommodate the aggregate resource constraint. The evolution of weights is derived on an individual basis by imposing each individual flow budget constraint separately ignoring the central planner.

*I am highly indebted to Professor Bernard Dumas for having provided me with a lot of insights and useful Mathematica codes. I am absolutely grateful to Professor Julien Hugonnier for great help. Also, I would like to thank Daniel Andrei and Alexandre Jeanneret for discussions on the topic. Correspondence: Swiss Finance Institute at EPFL, Quartier UNIL-Dorigny, Extranef 129, CH-1015 Lausanne, Switzerland; e-mail julien.cujean@epfl.ch.

[1] See Basak & Cuoco (1998), Guvenen & Kuruscu (2006), Heaton & Lucas (1996), Gomes & Michaelides (2006), Krusell & Smith (1997) or Krueger & Lustig (2007).

The first work achieving a proper central-planning approach is attributed to Grossman (1977): Grossman (1977) observes that, in an incomplete market setup, an individual utility maximization problem cannot be represented as a problem without random endowments but with initial capital increased by the expected discounted sum of future income. The reason is that the individuals are not able to perfectly coordinate their expenditures across states and time. As a result, Grossman (1977) requires a multi central planners approach and assigns a representative agent to each state of the economy.

Both Grossman (1977) and Cuoco & He (2001) approaches are global as one simultaneously solves for the equilibrium along all possible paths. Consequently, the dimension of the equilibrium computation is proportional to the number of states. Moreover, the path dependence induced by the endogenous component of state prices (say distribution of wealth or portfolio holdings in the primal problem) compromises the use of a recombining tree. For these reasons, these methods are expected to be most cumbersome for long horizon applications such as continuous time approximation.

In this perspective, a recursive approach involving a step-by-step computation of the equilibrium at each point in time is a way to bypass the mentioned issue. Cuoco & He (1994), in a continuous time setting, and Dumas & Lyasoff (2008), in a discrete time setting, propose a procedure that implements this purpose. Again, a state-dependent utility representative agent is used to obtain the equilibrium and, hence, the procedure is not centralized.

Reminiscent of Grossman (1977), Dumas & Maenhout (2003) introduce an incomplete market central planning approach based on the use of two selves. The idea is to propose a centralized version of Cuoco & He (1994): an additional self is taking care of the individual budget constraints and is conceived as the aggregation of the individual He & Pearson (1991) problems. The interaction between the two selves is described by a Nash game in which both planners cannot coordinate their moves with each other. Both selves accordingly articulated result into a central planning procedure which achieves the competitive equilibrium. As a consequence, the equilibrium requires a single recursion over the sole self value function instead of each individual financial wealth as in Cuoco & He (1994).

In the present note, one illustrates that this procedure fails to reproduce an incomplete market equilibrium when used recursively. More precisely, the Nash game regulating the interaction between the two parts of the representative agent cannot be solved along the sole equilibrium path but must be solved within the entire state space. For the exposition, one keeps away from infinite dimensional spaces in which the existence of an equilibrium is not guaranteed. When trading is continuous, market incompleteness may induce bubbles as highlighted by Heston, Loewenstein & Willard (2007) and Hugonnier (2007). Also, in terms of numerical computations, one has to determine nontrivial boundary conditions which do not appear when considering a tree. Accordingly, it is best to recast Dumas & Maenhout (2003) in a discrete time setup.

This note is organized as follows: Section 2 introduces notations and describes the economy in which Dumas & Maenhout (2003) is reformulated, Section 3 presents individual optimality and discusses recursivity issues, Section 4 illustrates the theoretical importance of the representative agent provided with two selves within a global setup and Section 5 proceeds with the approach shortcoming within a recursive setting and provides an illustration, finally Section 6 concludes. All equations that do not appear in the mentioned sections are reported in the appendix for the sake of clarity.

2 The Economy

In this section, one introduces the discrete time framework in which one recasts Dumas & Maenhout (2003).

2.1 Information Structure

Trading occurs at a finite number of dates $t \in \{0,1,...,T\} \subset \mathbb{N}_+$ where $T \geq 1$. Moreover, there is a finite number of states of the world $\omega \in \Omega$. Information is revealed through a partition of Ω given by $\mathbf{F} \triangleq \{\mathbf{F}_t : t = 0,1,...,T\}$ with $\mathbf{F}_0 \equiv \Omega$ and $\mathbf{F}_T \triangleq \{\{\omega\} : \omega \in \Omega\}$ being the discrete partition. Let $\mathcal{F}_t \triangleq \sigma(\mathbf{F}_t)$ be the $\sigma-$field generated by $\mathbf{F}_t$ and $\mathcal{F} \triangleq \{\mathcal{F}_t : t = 0,1,...,T\}$ be the corresponding filtration. The information structure $\mathbf{F}$ is represented by an event tree which one describes as follows: A tree is a collection of nodes s defined as the pair $s \triangleq (t, x_t)$ with $t \in \{0,1,...,T\}$, $x_t \in \mathbf{F}_t$ and

$$s \in \mathcal{S} \triangleq \cup_{\substack{t\in\{0,1,...,T\} \\ x_t \in \mathbf{F}_t}} (t, x_t).$$

Each node satisfies a precedence relation denoted $s' \succ s$ which is transitive and asymmetric[2]. The set of immediate successors to a node $s \equiv (t, x_t)$ is given by $S^+ \triangleq \cup_{x_{t+1}\in\mathbf{F}_{t+1}} (t+1, x_{t+1})$. More generally, one considers the set of all successors to node $s \equiv (t, x_t)$ as $S^{++} \triangleq \cup_{\substack{\tau\in\{t+1,...,T\} \\ x_\tau \in \mathbf{F}_\tau}} (\tau, x_\tau)$ with $S^+ \subset S^{++}$. Let one regard $o \triangleq (0, \Omega)$ as the initial node and $s_T \triangleq (T, \omega)$ as a terminal node. Let $\mathcal{S}_T \subset \mathcal{S}$ be the set of all terminal nodes and accordingly one defines $\mathcal{S}^+ \triangleq \mathcal{S} \backslash \{\mathcal{S}_T\}$ and $\mathcal{S}^- \triangleq \mathcal{S} \backslash \{o\}$. Finally, let $m \triangleq \#(S^+)$ be the branching number and $K \triangleq \#(\Omega)$ be the total number of nodes. Hence, any $\mathbb{R}^N-$valued stochastic process $\{\mathbf{X}(s) : s \in \mathcal{S}\}$ adapted to the filtration $\mathcal{F}$ is viewed as a function $X : \mathcal{S} \to \mathbb{R}^{N\times K}$.

2.2 Agents

One considers an exchange economy with one consumption good. There is a possibly large but finite number of agents $I+1 \geq 2$ where the $I+1^{th}$ agent will be used as a normative individual. Each agent $i \in \mathcal{I} \triangleq \{1,...,I+1\}$ trades competitively and is endowed with an initial financial wealth $F_i(o) \in \mathbb{R}_+$ with $\sum_{i\in\mathcal{I}} F_i(o) = 0$ which is measured in terms of units of the consumption good taken as the *numéraire*. Agent i receives an adapted endowment flow $\{\delta_i(s) : s \in \mathcal{S}\}$ with $\delta_i \in \{F_i(o)\} \times \mathbb{R}^{K-1}_{++}$, and may be thought of as an exogenous adapted income process. Denote $\delta(s)$ the aggregate endowment $\sum_{i\in\mathcal{I}} \delta_i(s)$ $\forall s \in \mathcal{S}$. The set of each agent's consumption is the set of real-valued, strictly positive, adapted processes and may be written $\{c_i(s) : s \in \mathcal{S}\}$ with $c_i \in \mathbb{R}^K_{++}$. The beliefs of agent i are represented by a strictly positive probability measure $\mathbb{P}_i$ defined on $(\Omega, \mathcal{F})$ with $\mathbb{P}_i : \mathcal{F}_t \to [0,1]$ such that for any node $s = (t, x_t)$, the transition probability associated with any of its successors $s' = (\tau, x_\tau) \in S^{++}$ is $\mathbb{P}_i(s'|s) \equiv \mathbb{P}_i(x_\tau|x_t) \equiv \frac{\sum_{\omega\in x_\tau \cap x_t} \mathbb{P}_i(\omega)}{\sum_{\omega\in x_t} \mathbb{P}_i(\omega)} \triangleq \pi_i(s'|s)$. One normalizes (with respect to

[2]Because the tree only accommodates for the exogenous endowment processes and that the latter are Markovian, in a dynamic setting a recombining tree may be used. Whereas, the static equilibrium requires the computation of optimal strategies along all possible trajectories. Since the presented equilibrium is of a recursive nature, the relations between nodes are not restricted to single precedence. This point will be made in further details when discussing recursivity in the next section.

agent $I+1$'s beliefs) this probability so as to consider a single probability measure $\mathbb{P}$ for all agents. Hence, $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space one is led to deal with.

Moreover, agent i's life-time expected preferences are represented with a time- and state- additive utility function $U_i : \mathbb{R}^K_{++} \to \mathbb{R}$ where

$$U_i(c_i) \triangleq \sum_{s \in \mathcal{S}} \pi(s) u_i(c_i(s), s).$$

The functions $u_i(., s)$ are *assumed* to be increasing, strictly concave, twice differentiable on $\mathbb{R}_{++}$ for each $s \in \mathcal{S}$ and to satisfy the Inada conditions. These assumptions aim to make sure that $c_i \in \mathbb{R}^K_{++}$ because $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$f_i(., s) \triangleq \left(\frac{\partial}{\partial c} u_i(c_i(s), s) \right)^{-1}$$

exists and guarantees an interior solution.

2.3 Financial Assets and Portfolio Strategies

There are $N+1$ securities one of which is assumed to be riskless and, by premise, $N+1 < m \; \forall s \in \mathcal{S}^-$ (i.e. markets are dynamically incomplete). Each security $n \in \mathcal{N} \triangleq \{1, ..., N+1\}$ is identified by a given adapted dividend process $\{\iota_n(s) : s \in \mathcal{S}\}$ with $\iota_n \in \mathbb{R}^K$ and $\iota \in \mathbb{R}^{(N+1) \times K}$. Denote the adapted price[3] process of security n by $\{P_n(s) : s \in \mathcal{S}\}$ such that $\mathbf{P} \in \mathbb{R}^{N+1}$. Individuals freely invest in each asset whose total net supply is zero[4]. Accordingly, $\theta_i \triangleq \left[\alpha_i, \{\theta_{i,n}\}_{n \in \mathcal{N} \setminus \{N+1\}} \right]^\top$ is the portfolio choice process of agent i where $\theta_{i,n}(s)$ denotes the number of shares of security n in the portfolio established at node s and carried into the nodes in S^+ and, similarly, $\alpha(s)$ is the number of units invested in the riskless asset at node s and carried into nodes S^+. Moreover, define the trading space as $\Theta \triangleq \mathbb{R}^{(N+1) \times (K - \#(\mathcal{S}_T))}$ such that $\theta_i \in \Theta$. Finally, as already mentioned,

$$\sum_{i \in \mathcal{I}} \alpha_i(s) = 0 \quad \forall s \in \mathcal{S}^+ \tag{1}$$

$$\sum_{i \in \mathcal{I}} \theta_{i,n}(s) = 0 \quad \forall s \in \mathcal{S}^+, \; n \in \mathcal{N} \setminus \{N+1\}. \tag{2}$$

Therefore, denote the primitives of the above economy as

$$\mathcal{E} \triangleq \left((\Omega, \mathcal{F}, \mathbb{P}), \{U_i, \delta_i\}_{i \in \mathcal{I}}, \iota \right).$$

[3] In the next section, the dual approach will be used such that the vector of prices $\mathbf{P}$ does not have to be specified *ab initio*. Moreover, at this point the riskless bond may be either conceived as a long run asset or as a sequence of one-period (short-lived) bonds.

[4] This is without loss of generality since the payoffs may include endowments in which case assets are in positive supply.

3 Individual Optimality

3.1 Minimax Individual Programs

One formulates optimisation programs using the well-known *dual* approach[5] also referred to as the *martingale representation technique:* it was introduced by Pliska (1986), Cox & Huang (1989,1991) and Karatzas, Lehoczky & Shreve (1987) within a complete financial markets setting and by He & Pearson (1991a,b) and Karatzas, Lehoczky, Shreve & Xu (1991) within an incomplete financial markets framework. One may refer to Hugonnier & Kramkov (2003) for a detailed characterization of the feasibility of the latter approach for terminal wealth with intermediate random endowments problem as well as Karatzas & Zitkovic (2007) when there are interim consumptions. Adopting this procedure permits to avoid having to postulate price processes *ab initio.* By the same token, it allows to bypass the *tâtonement* approach and, thus, facilitates the computation of the equilibrium. In this respect, this procedure is fundamental for the construction of a representative individual.

Agent i maximizes his expected life-time utility

$$\sup_{\{c_i \in \mathbb{R}_{++}\}} \inf_{\{\phi_i \in \mathbb{R}_{++}\}} U_i(c_i) \tag{6}$$

subject to his life-time budget constraint

$$\sum_{s \in \mathcal{S}^-} \pi(s)\phi_i(s)(c_i(s) - \delta_i(s)) \leq F_i(o) \ , \phi_i(o) \equiv 1 \tag{7}$$

and the pricing restriction

$$\phi_i(o)P_n(o) = \sum_{s \in \mathcal{S}^-} \pi(s)\phi_i(s)\iota_n(s) \ \forall \ n \in \mathcal{N} \tag{8}$$

where $F_i(o)$ is agent i's initial financial wealth. ϕ_i represents the marginal rate of substitution of agent i between consumption at node s and $s \in o$. Put differently, $\phi_i(s)$ is investor i's shadow Arrow-Debreu price of a claim paying only in state s. Notice that agent i chooses ϕ_i to saturate his budget constraint in such a way that, when compared across investors, they all face the same quoted prices. (8) is denoted the kernel condition because it restricts the shadow prices to lie within a subspace defined by the linear valuation mapping.

[5] In the primal approach, agent i solves

$$\sup_{\{c_i \in \mathbb{R}_{++}\}, \{\theta_i \in \Theta\}} U_i(c_i) \tag{3}$$

$$s.t. \ c_i(s) + \sum_{n \in \mathcal{N}} \theta_{i,n}(s)P_n(s) \leq F_i(s) + \delta_i(s) \ \ \forall s \in \mathcal{S}^+. \tag{4}$$

The Lagrangean of the problem is written as

$$\inf_{\{\mu_i, \phi_i \in \mathbb{R}_{++}\}} \sup_{\{c_i \in \mathbb{R}_{++}\}, \{\theta_i \in \Theta\}} \left\{ U_i(c_i) + \mu_i \sum_{s \in \mathcal{S}^+} \pi(s)\phi_i(s) \left[F_i(s) + \delta_i(s) - \sum_{n \in \mathcal{N}} \theta_{i,n}(s)P_n(s) - c_i(s) \right] \right\}. \tag{5}$$

Accordingly, the Lagrangean for that problem is

$$J_i\left(\mu_i\phi_i(s),\delta_i(s),s\right) \triangleq \sup_{\{c_i\in\mathbb{R}_{++}\}} \inf_{\{\mu_i,\phi_i\in\mathbb{R}_{++}\}} \sup_{\{\theta_i\in\mathbb{R}^N\}} U_i(c_i) \tag{9}$$
$$+\mu_i\left(F_i(o)-\sum_{s\in\mathcal{S}^-}\pi(s)\phi_i(s)(c_i(s)-\delta_i(s))\right)$$
$$+\mu_i\sum_{n\in\mathcal{N}}\left(\sum_{s\in\mathcal{S}^+}\pi(s)\theta_{i,n}(s)\left(\sum_{s'\in S^{++}}\pi(s'|s)\phi_i(s')\iota_n(s')-\phi_i(s)P_n(s)\right)\right)$$

where $\mu_i \times \theta_{i,n}(s)$ is the Lagrange multiplier related to the kernel condition at node $s \in \mathcal{S}^+$. One may readily observe that solving this program will deliver three first order conditions representing 1) the optimal consumption, 2) the primal flow budget constraint and 3) the pricing restriction.

Maximizing (9) with respect to c_i, one obtains the dual shadow state-price problem

$$\widehat{H}(\mu_i) \triangleq \inf_{\{\mu_i,\phi_i\in\mathbb{R}_{++}\}} \sup_{\{\theta_i\in\mathbb{R}^N\}} \widehat{I}(\phi_i) \triangleq \sum_{s\in\mathcal{S}}\pi(s)\widehat{u}_i(\mu_i\phi_i(s),s) \tag{10}$$
$$+\mu_i\left(F_i(o)+\sum_{s\in\mathcal{S}}\pi(s)\phi_i(s)\delta_i(s)\right)$$
$$+\mu_i\sum_{n\in\mathcal{N}}\left(\sum_{s\in\mathcal{S}^+}\pi(s)\theta_{i,n}(s)\left(\sum_{s'\in S^{++}}\pi(s'|s)\phi_i(s')\iota_n(s')-\phi_i(s)P_n(s)\right)\right)$$

where

$$\widehat{u}_i(\mu_i\phi_i(s),s) \triangleq \sup_{\{c_i\in\mathbb{R}_{++}\}} u_i(c_i(s),s)-\mu_i\phi_i(s)c_i(s) \tag{11}$$

denotes the convex conjugate of $-u_i(-c_i(s),s)$.

As pointed out by Cuoco (1997) in a continuous-time setting, interim consumptions and random endowments generally make the map $\phi_i \rightarrow$(10) not convex. For the problem only involving terminal wealth maximization with terminal random endowment, Kramkov & Schachermayer (1999) have given sufficient conditions on the utility function for this saddle point to exist. Their approach enlarges the usual dual space to $(\mathbb{L}^\infty)^*$ which is a space of finitely-additive measures and assumes boundedness of endowment. By considering initial capital as well as number of units of random endowments as variables of the optimization problem, Hugonnier & Kramkov (2003) show that the dual problem does not require the use of finitely-additive measures and that endowments are bounded. Crucially, Karatzas & Zitkovic (2007), using the same tools as in Kramkov & Schachermayer (1999), show that the above map is convex provided that the utility function is *reasonably* elastic. Fortunately, this fundamental issue does not apply to discrete time and was already solved by He & Pearson (1991 a) and Shirakawa (1994): If one denotes Φ the set of dual admissible control processes, it is trivial to check that it is a nonempty convex cone since the dual variables are defined over the strictly positive part of the real line. One may define a modulus subset $\Phi_\varphi \subset \Phi$ by

$$\Phi_\varphi \equiv \{\phi_i \in \Phi : \phi_i(o) = \varphi\}$$

for $\varphi \in \mathbb{R}_{++}$.

Since Φ is a convex cone, $\Phi_{\lambda\varphi_1+(1-\lambda)\varphi_2} = \lambda\Phi_{\varphi_1} + (1-\lambda)\Phi_{\varphi_2}$ for $\lambda \in [0,1]$ and $\varphi_1, \varphi_2 \in \mathbb{R}_{++}$. Since $\widehat{I}$ is convex, one obtains

$$\begin{aligned}
\widehat{H}\left(\lambda\varphi_1 + (1-\lambda)\varphi_2\right) &= \inf_{\phi \in \Phi_{\lambda\varphi_1+(1-\lambda)\varphi_2}} \widehat{I}(\phi_i) \\
&= \inf_{\phi_1 \in \Phi_{\varphi_1}, \phi_2 \in \Phi_{\varphi_2}} \widehat{I}\left(\lambda\phi_1 + (1-\lambda)\phi_2\right) \\
&\leq \inf_{\phi_1 \in \Phi_{\varphi_1}, \phi_2 \in \Phi_{\varphi_2}} \left\{\lambda\widehat{I}(\phi_1) + (1-\lambda)\widehat{I}(\phi_2)\right\} \\
&= \lambda \inf_{\phi_1 \in \Phi_{\varphi_1}} \widehat{I}(\phi_1) + (1-\lambda) \inf_{\phi_2 \in \Phi_{\varphi_2}} \widehat{I}(\phi_2) \\
&= \lambda\widehat{H}\left(\varphi_1\right) + (1-\lambda)\widehat{H}\left(\varphi_2\right).
\end{aligned}$$

Hence, (10) is convex. One may then refer to Shirakawa (1994) for the final claim that a saddle point in (10) exists.

3.2 Recursivity

The dynamic programming formulation of (9) is given by

$$\begin{aligned}
J_i(\mu_i\phi_i(s), \delta_i(s), s) \triangleq\ & \sup_{\{c_i \in \mathbb{R}_{++}\}} \inf_{\{\mu_i, \phi_i \in \mathbb{R}_{++}\}} \sup_{\{\theta_i \in \mathbb{R}^N\}} \sum_{s' \in S^+} \pi(s'|s) u_i(c_i(s')) \\
& + \mu_i \left(F_i(s) - \sum_{s' \in S^+} \pi(s'|s)\phi_i(s')(c_i(s') - \delta_i(s'))\right) \\
& + \mu_i \sum_{n \in \mathcal{N}} \left(\theta_{i,n}(s) \left(\sum_{s' \in S^+} \pi(s'|s)\phi_i(s')(P_n(s') + \iota_n(s')) - \phi_i(s)P_n(s)\right)\right) \\
& + \sum_{s' \in S^+} \pi(s'|s) J_i(\mu_i\phi_i(s'), \delta_i(s'), s').
\end{aligned} \tag{12}$$

(12) is just the Bellman equation associated with (9) and delivers three first order conditions given by

$$\frac{\partial}{\partial c_i} u_i(c_i(s')) = \mu_i\phi_i(s'), \tag{13}$$

$$-c_i(s') + \delta_i(s') + \sum_{n \in \mathcal{N}} \theta_{i,n}(s)(P_n(s') + \iota_n(s')) + \frac{\partial}{\partial \mu_i\phi_i(s')} J_i(\mu_i\phi_i(s'), \delta_i(s'), s') = 0 \tag{14}$$

and

$$\sum_{s' \in S^+} \pi(s'|s)\phi_i(s')(P_n(s') + \iota_n(s')) - \phi_i(s)P_n(s) = 0. \tag{15}$$

Finally, the envelope condition implies

$$\frac{\partial}{\partial \mu_i\phi_i(s)} J_i(\mu_i\phi_i(s), \delta_i(s), s) = -\sum_{n \in \mathcal{N}} \theta_{i,n}(s)P_n(s) \equiv -F_i(s). \tag{16}$$

(16) as well as optimal consumption described by (13) are substituted into (14) to obtain

$$f_i(\mu_i \phi_i(s')) + F_i(s') = \delta_i(s') + \sum_{n \in \mathcal{N}} \theta_{i,n}(s)(P_n(s') + \iota_n(s')) \tag{17}$$

$\forall s \in \mathcal{S}^-$.

Although the system made of the budget flow constraint (17) and the pricing restriction (15) may well be solved for $\{\theta_n(s) : n \in \mathcal{N}\}$ and $\{\phi_i(s') : s' \in S^+\}$ recursively, one prefers to change the formulation of the problem. The reason is twofold: First, the construction of the central planner requires that ϕ_i contains separate exogenous and endogenous parts. Accordingly, one considers the following change of variables

$$\phi_i(s) \equiv \xi(s) \times \eta_i(s) \tag{18}$$

where $(\xi, \eta_i) \in \{1\} \times \{1\} \times \mathbb{R}^{K-1}_{++} \times \mathbb{R}^{K-1}_{++}$. η_i and ξ constitute the endogenous and exogenous state variable of the shadow price respectively.

Second, the backward-forward aspect of the problem accrues for a new structure of the endogenous variables: In a recursive setting, one is led to use dynamic programming which induces backward induction. But, on the other hand, $\eta_i(s)$ is determined at node s in a forward fashion given an initial condition. Therefore, when solving the program at node $s' \in S^+$, $\eta_i(s)$ is exogenous and will only be endogenously determined one step ahead. For this reason, it is attributed a law of motion from node s to nodes $s' \in S^+$

$$\eta_i(s') = \eta_i(s)(1 - \nu_i(s')) \ \forall s \in \mathcal{S}^+. \tag{19}$$

The rationale for (19) is to keep track of future state variable by considering its variations. Whereas, $\{\xi(s) : s \in \mathcal{S}\}$ may just be a jump process because it is purely exogenous.

One notes that this decomposition may be achieved in an infinite number of ways: in order to pin down choices, one sets $\nu_{I+1}(s) \equiv 0 \ \forall s \in \mathcal{S}^-$. This normalization is similar to the spanning condition used in the continuous time literature.

3.3 Existence

In an incomplete market setting, exogenous shocks do not constitute a sufficient statistic for the future evolution of the system. The state space has to be enriched with endogenous variables such as the distribution of wealth or portfolio holdings in order to make the equilibrium computation feasible. Kubler & Schmedders (2002) consider examples in which adding such endogenous variables is not sufficient to guarantee the existence of an equilibrium. Krebs (2004) further argues that unless consumptions are made parts of the state space, recursive equilibria do not exist. However, Krebs' argumentation requires continuity of consumption streams which may not always be verified. Duffie, Geanakoplos, Mas-Collel & McLennan (1994) prove the existence of generic ergodic markovian equilibria[6].

This note aims to illustrate how Dumas & Maenhout (2003) fails to replicate a recursive incomplete market central planning. In this perspective, existence of such an equilibrium is obviously not in the scope of the present discussion.

[6] One may also refer to Duffie & Shafer (1985), Duffie & Shafer (1986).

4 Equilibrium & Representative Agent: Global Formulation

In this section, one argues in favor of a representative agent that would use two central planners. To that purpose, one discusses several related works and emphasizes the improvement brought by the latter approach.

4.1 Cuoco & He (2001): A Starter

One briefly presents the Cuoco & He (2001) procedure recast in the present discrete framework and shows that it does not require a representative agent. This will serve as an illustration of the theoretical importance of the central planning approach developed in Dumas & Maenhout (2003). Consider the standard representative agent except for allowing its Negishi weights to be stochastic: using the notations previously introduced, this agent would solve at node o

$$\begin{aligned} & \sup_{\{\sum_{i\in\mathcal{I}} c_i(s)=\delta(s)\}} \sum_{s\in S} \pi(s) \sum_{i\in\mathcal{I}} \frac{1}{\mu_i \eta_i(s)} u_i(c_i(s)) \\ \equiv \; & \sup_{\{\sum_{i\in\mathcal{I}} c_i(s)=\delta(s)\}} \sum_{s\in S} \pi(s) \sum_{i\in\mathcal{I}} \widetilde{u}_i(c_i(s), \mu_i \eta_i(s)). \end{aligned} \tag{20}$$

Cuoco & He (1994) normalizes with respect to agent 1 (in this case agent $I+1$) and obtain

$$\begin{aligned} & \sup_{\{\sum_{i\in\mathcal{I}} c_i(s)=\delta(s)\}} \sum_{s\in S} \pi(s) \sum_{i\in\mathcal{I}} \frac{\mu_{I+1}\eta_{I+1}(s)}{\mu_i \eta_i(s)} u_i(c_i(s)) \\ \triangleq \; & \sup_{\{\sum_{i\in\mathcal{I}} c_i(s)=\delta(s)\}} \sum_{s\in S} \pi(s) \sum_{i\in\mathcal{I}} \lambda_i(s) u_i(c_i(s)) \end{aligned} \tag{21}$$

where $\lambda_{I+1}(s) \equiv 1 \; \forall s \in \mathcal{S}$.

From (21), one derives the first order conditions for each agent i and at each node s

$$\frac{\partial}{\partial c_i} u_i(c_i(s)) = \frac{\xi(s)}{\lambda_i(s)} \qquad \forall i \in \mathcal{I}, \;\; s \in \mathcal{S} \tag{22}$$

and the aggregate resource constraint where $\xi(s)$ is the multiplier of the aggregate resource constraint at node s. (22) only serves to determine each agent optimal consumption as

$$c_i^*(s) = f_i\left(\frac{\xi(s)}{\lambda_i(s)}, s\right) \qquad \forall i \in \mathcal{I}, \;\; s \in \mathcal{S}. \tag{23}$$

Hence, merging the aggregate resource constraint and (23), the representative agent only accomodates at each node s

$$\sum_{i\in\mathcal{I}} f_i\left(\frac{\xi(s)}{\lambda_i(s)}, s\right) = \delta(s) \qquad \forall s \in \mathcal{S}. \tag{24}$$

If one had known every $\mu_i \eta_i$ (i.e. the weights λ_i), one would have been able to solve for ξ using (24) as in a complete market setting. However, the evolution of the weights is driven by $\eta_i(1-\nu_i)$ such that the representative agent does not control it. These variables are determined using the individual[7] programs (9). Those conditions at each node $s \in \mathcal{S}^+$ are given by (17) and (15) reexpressed here with

[7]These programs are given by (33)-(34) in Cuoco & He (1994).

the new formulation

$$f_i\left(\frac{\xi(s)}{\lambda_i(s)}, s\right) + F_i(s') = \theta_i(s)^\top \left(\iota(s') + \mathbf{P}(s')\right) + \delta_i(s') \quad \forall i \in \mathcal{I},\ s \in \mathcal{S} \tag{25}$$

and

$$\eta_i(s)\xi(s)P_n(s) = \sum_{s' \in S^{++}} \pi(s'|s)\eta_i(s')\xi(s')\iota_n(s') \quad \forall i \in \mathcal{I},\ s \in \mathcal{S} \tag{26}$$

where (25) is individual i flow budget constraint and (26) is the kernel condition. The discounted financial wealth is defined as

$$\xi(s) \times F_i(\mu_i\eta_i(s), \delta_i, s) \triangleq \sum_{s' \in S^{++}} \pi(s'|s)\xi(s')(1-\nu_i(s'))(c_i(s') - \delta_i(s')). \tag{27}$$

The equilibrium requires to solve (24)-(26) simultaneously at each node s. It is checked that the central planner was useless in determining the evolution of the weights such that it could have been ignored. Indeed, the equilibrium computation only involves optimality for every individual as well as the aggregate resource constraint.

4.2 Magill & Quinzii (1996): The Representative Agent Shortcoming

In Basak & Cuoco (1998), the representative agent approach is described as a *weaker* central planning approach. This is true for two reasons: First, as highlighted in the previous subsection, the representative agent cannot achieve a fully centralized equilibrium. Second, an incomplete market equilibrium cannot be regarded as a Pareto optimum. Magill & Quinzii (1996, chapter IV) present an example in which they show that a single representative agent is unable to achieve an equilibrium in the Pareto Optimality sense. The equilibrium is not a Pareto optimum even under the constraint of marketability of consumption plans. Indeed, the classical result of the representative agent existence is due to the first welfare theorem which does not apply for a large class of economies within an incomplete market setting. In this respect, one has to be careful not to attribute any welfare interpretation to the representative agent one aims to construct. Notice that, for a subclass of economies, constrained optimality still holds: this will be the case whenever one of the following five conditions are met: 1) markets are complete, 2) $T = 1$, 3) agents have identical homothetic preferences, 4) it is optimal for agents not to trade and 5) there is a single agent in the economy.

As an illustration, consider an economy with two agents having logarithmic preferences and patience parameters $(\beta^1, \beta^2) \equiv \left(\frac{1}{2}, \frac{1}{3}\right)$ respectively. The exogenous event tree for the aggregate state Y_s has three dates $t \in \{0, 1, 2\}$, uncertainty is resolved at time $t = 1$ where two states of nature $s \in \{1, 2\}$ may occur with probability $\pi \equiv \frac{1}{2}$. The information partitions are thus

$$\mathbf{F}_0 = \{1, 2\},\ \mathbf{F}_1 = \{\{1\}, \{2\}\} \text{ and } \mathbf{F}_2 = \{\{1\}, \{2\}\}.$$

The exogenous endowment processes obey

$$\begin{aligned}
\delta_1 &= (\delta_1(Y_0), \delta_1(Y_1), \delta_1(Y_2), \delta_1(Y_{12}), \delta_1(Y_{22})) \equiv (4, 0, 6, 6, 6), \\
\delta_2 &= (\delta_2(Y_0), \delta_2(Y_1), \delta_2(Y_2), \delta_2(Y_{12}), \delta_2(Y_{22})) \equiv (9, 8, 0, 8, 8).
\end{aligned}$$

Finally, in each period, there is only one financial asset, a short-lived bond freely traded. Notice that markets are incomplete since agents are unable to hedge their endowment process because a risky asset whose payoffs are contingent on each state realization fails to exist.

The life-time utility of agent i is given by

$$U_i(c_i) = \log c_i(Y_0) + \beta_i \left(\begin{array}{c} \frac{1}{2}\left(\log c_i(Y_1) + \beta_i \log c_i(Y_{12})\right) \\ +\frac{1}{2}\left(\log c_i(Y_2) + \beta_i \log c_i(Y_{22})\right) \end{array} \right). \tag{28}$$

At time $t = 1$ in state $s \in \{1, 2\}$, agent i's intertemporal budget constraint obeys

$$c_i(Y_s) + \frac{\xi(Y_{s2})}{\xi(Y_s)} c_i(Y_{s2}) = \delta_i(Y_s) + \theta_i(Y_0) + \frac{\xi(Y_{s2})}{\xi(Y_s)} \delta_i(Y_{s2}). \tag{29}$$

Solving agent i's intertemporal problem using (28) and (29), it is straightforward to see that the optimal consumption plans obey

$$\begin{aligned} c_i(Y_s) &= \frac{1}{1+\beta_i}\left(\delta_i(Y_s) + \theta_i(Y_0) + \frac{\xi(Y_{s2})}{\xi(Y_s)}\delta_i(Y_{s2})\right), \\ c_i(Y_{s2}) &= \frac{\beta_i}{1+\beta_i}\left(\frac{\xi(Y_s)}{\xi(Y_{s2})}\left(\delta_i(Y_s) + \theta_i(Y_0)\right) + \delta_i(Y_{s2})\right) \end{aligned}$$

and that equilibrium prices are described by

$$\frac{\xi(Y_{s2})}{\xi(Y_s)} = \frac{\frac{\beta_1}{1+\beta_1}\left(\delta_1(Y_s) + \theta_1(Y_0)\right) + \frac{\beta_2}{1+\beta_2}\left(\delta_2(Y_s) + \theta_2(Y_0)\right)}{\frac{1}{1+\beta_1}\delta_1(Y_{s2}) + \frac{1}{1+\beta_2}\delta_2(Y_{s2})}.$$

Consider changing the date 0 portfolios by $\Delta\theta_i(Y_0)$ with $\Delta\theta_1(Y_0) \equiv \Delta\theta(Y_0)$ and $\Delta\theta_2(Y_0) = -\Delta\theta(Y_0)$. The relation above indicates that the change in prices would amount to

$$\Delta\left(\frac{\xi(Y_{s2})}{\xi(Y_s)}\right) = \frac{\left(\frac{\beta_1}{1+\beta_1} - \frac{\beta_2}{1+\beta_2}\right)}{\frac{1}{1+\beta_1}\delta_1(Y_{s2}) + \frac{1}{1+\beta_2}\delta_2(Y_{s2})}\Delta\theta(Y_0).$$

Imposing no resource transfer when changing initial portfolio positions, the change $\Delta\theta(Y_0)$ induces a change in agent i utility defined over $t = 1, 2$ given by

$$\frac{\Delta u_i}{\mu_i} = \sum_{s\in\{1,2\}} \pi(Y_s)\eta_i(Y_s)\xi(Y_s)\Delta\left(\frac{\xi(Y_{s2})}{\xi(Y_s)}\right)\left(-\theta_i(Y_s)\right)$$

yielding in this particular case $\frac{\Delta u_1}{\mu_1} = \frac{1}{32}\Delta\theta(Y_0)$ and $\frac{\Delta u_2}{\mu_2} = \frac{1}{18}\Delta\theta(Y_0)$. Hence, if one chooses $\Delta\theta(Y_0) > 0$, one increases the welfare of each agent such that both individuals are better off.

4.3 Grossman (1977): Multiple Central Planners Approach

Given the absence of Pareto optimality in an incomplete market setting, Grossman (1977) uses the weaker concept of *Social Nash Optimality.* The set of Social Nash Optima characterizes a larger set of optimal allocations such that Pareto optimality is a subset of the latter. Grossman describes a multi-good economy with two dates $t \in \{0, 1\}$ and where n states of nature may occur at date $t = 1$.

In order to be in accordance with the previous setting and since the number of goods is not crucial to the presentation, one considers the special case of a single good economy.

Consider the two consumption-investment problems

$$\begin{aligned} &\sup_{\{c_i(o),\theta_i(o),c_i(s)\}} U_i(c_i(o), c_i(s)) \\ &s.t.\ c_i(o) + \sum_{s\in S^+} \pi(s)\xi(s)c_i(s) + \sum_{n\in\mathcal{N}} \theta_{i,n}(o)P_n(o) \\ \leq\ &\delta_i(o) + \sum_{s\in S^+} \pi(s)\xi(s)\left(\delta_i(s) + \sum_{n\in\mathcal{N}} \theta_{i,n}(o)\iota_n(s)\right) \end{aligned} \tag{30}$$

and

$$\begin{aligned} &\sup_{\{c_i(o),\theta_i(o)\}} U_i(c_i(o), c_i(s)) \\ &s.t.\ c_i(o) + \sum_{n\in\mathcal{N}} \theta_{i,n}(o)P_n(o) \leq \delta_i(o) \\ &\sup_{\{c_i(s)\}} U_i(c_i(o), c_i(s)) \quad \forall s \in S^+ \\ &s.t.\ \xi(s)c_i(s) \leq \xi(s)\left(\delta_i(s) + \sum_{n\in\mathcal{N}} \theta_{i,n}(o)\iota_n(s)\right) \end{aligned} \tag{31}$$

where (31) is solved simultaneously, i.e. the optimization in each state is operated fixing the other states optimal allocations.

Grossman (1977) makes the crucial observation that, in the complete market case, (30) and (31) are perfectly equivalent. However, in the incomplete market setup, only (31) may be considered. Indeed, (31) cannot be expressed as an optimal investment problem without random endowments but with an augmented initial capital. The reason is that agents are not able to perfectly coordinate their expenditures across time and states due to the insufficient security span. That is, random endowments cannot be perfectly replicated by a portfolio of the available assets. This observation justifies the notion of an incomplete coordination equilibrium in which the optimization is performed in each state separately while fixing other decisions across time and states.

It is then clear that a central planning procedure requires as many central planners as there are states in the economy plus one for the date $t = 0$ allocation problem. Grossman (1977) shows that a Social Nash Optimum may be achieved through the use of $n+1$ poorly coordinated central planners acting as a regular complete market auctioneer except for being their own master in a single state only and letting the other ones decide in the remaining states.

4.4 Dumas & Maenhout (2003): Two Selves Central Planner

Grossman (1977) central planning approach contains the fundamental idea of incomplete coordination. It is now of practical interest to obtain a central planning procedure that lies between Grossman (1977)'s $n+1$ central planners and Cuoco & He (2001) imperfect representative agent. In this respect, one follows Dumas & Maenhout (2003) and considers the two following definitions of an equilibrium in $\mathcal{E}$.

Definition 1. *A competitive market equilibrium is a set of processes* $\{\{c_i\}, \{\theta_i\}\}$ *and price processes* $\{\mathbf{P}\}$ *such that, for each individual* i, $\{c_i\}$ *and* $\{\theta_i\}$ *are the optimized arguments of (3) subject to (4) and such that the market clearing conditions (1) and (2) hold.*

Suppose that a competitive market equilibrium exists, then one considers the following definition of a sub-equilibrium

Definition 2. *A competitive market "sub-equilibrium" is a set of processes* $\{\{c_i\}, \{\nu_i\}, \xi\}$*, in which, for each individual* i, $\{c_i\}, \{\nu_i\}$ *are the optimizing argument of (6), and which are such that the aggregate resource constraint holds.*

One is looking for a central-planning problem that generates a sub-equilibrium. As highlighted in the previous subsections, a single representative agent is only able to achieve a fake central planning equilibrium because each agent specific flow budget constraint has to hold on a separate basis. One requires an additional planner that makes sure the individual equations hold. Reminiscent[8] of Grossman (1977), the central planner one is interested in has two uncoordinated selves defined by the two *pseudo welfare functions*

$$\sup_{\{c_i \in \mathbb{R}^K_{++}\}} \inf_{\{\nu_i\}} \left\{ \begin{array}{c} \sum_{i\in\mathcal{I}} U_i(c_i) \\ -\sum_{i\in\mathcal{I}} \mu_i \sum_{s\in\mathcal{S}} \pi(s)\eta_i(s)\xi(s)(c_i(s)-\delta_i(s)) \end{array} \right\} \tag{32}$$

for Self 1 and

$$\sup_{\{c_i \in \mathbb{R}_{++}\}} \left\{ \begin{array}{c} \sum_{i\in\mathcal{I}} \sum_{s\in\mathcal{S}} \pi(s) \frac{1}{\mu_i \eta_i(s)} u_i(c_i(s), s) \\ + \inf_{\{\xi \in \mathbb{R}_+\}} \left[-\sum_{s\in\mathcal{S}} \pi(s) \sum_{i\in\mathcal{I}} \xi(s)(c_i(s)-\delta_i(s)) \right] \end{array} \right\} \tag{33}$$

for Self 2. Notice that $\{\nu_i\}$ is chosen such that the pricing restriction (7) holds in all states. One incorporates this constraint by attributing the Lagrange multiplier $\mu_i \times \theta_{i,n}(s)$ to (8) in the same fashion as in (9). One checks that Self 1 (i.e. (32)) works as the partial-equilibrium agent in He & Pearson (1991) by making sure each individual budget constraint is satisfied. Self 2 (i.e. (33)) almost acts as a complete market auctioneer imposing the aggregate resource constraint to hold. In other words, one just adds another central planner (32) to Cuoco & He (1994) representative agent (33).

Deriving the first order conditions for (32) and (33), one easily checks that both selves agree on the optimal consumption choice dictated by (23), Self 1 accommodates (24) and Self 2 imposes (25) and (26) to hold at each node. Grossman (1977)'s incomplete coordination concept particularizes as follows: (32) takes the whole trajectory $\{\xi(s) : s \in \mathcal{S}\}$ as given and (33) holds the whole path $\{\eta_i(s) : s \in \mathcal{S}\}$ as fixed. Put differently, since the system of equations described by (24), (25) and (26) is solved globally, i.e. for all nodes simultaneously, both selves cannot coordinate their moves across states and time. This interaction between the two parts of the representative individual describe a global Nash game in which each player takes the other's decision as given. The nature of the game is, however, trivial in a static framework because of the simultaneous aspect of the problem. It becomes more involved when one considers a dynamic formulation: this will be discussed in the next section.

Finally, one should crucially keep in mind that the central planner described above does not achieve a Pareto optimum in $\mathcal{E}$ and, thus, carries no welfare meaning. As already illustrated by Magill

[8] The methodology considerably differs because one only needs a two selves central planner when Grossman (1997) uses as many central planners as there are states plus one.

& Quinzii (1996) example and characterized by Grossman (1977), if one desires to get an interpretation of the central planner in terms of optimality, one should consider a larger class of optimal allocations which includes Pareto optimality as a special case.

Notice, as an illustration, that the equilibrium consumption allocation described by Magill & Quinzii (1996)

$$\begin{aligned} c_1 &= (c_1(Y_0), c_1(Y_1), c_1(Y_2), c_1(Y_{12}), c_1(Y_{22})) \equiv (4, 0.8, 4.8, 2, 12) \\ c_2 &= (c_2(Y_0), c_2(Y_1), c_2(Y_2), c_2(Y_{12}), c_2(Y_{22})) \equiv (9, 7.2, 1.2, 12, 2) \end{aligned}$$

along with the values (agent 2 is the normative individual)

$$\begin{aligned} \xi &= (\xi(Y_0), \xi(Y_1), \xi(Y_2), \xi(Y_{12}), \xi(Y_{22})) = (1, \frac{5}{12}, \frac{5}{2}, \frac{1}{12}, \frac{1}{2}), \\ \nu_1 &= (\nu_1(Y_0), \nu_1(Y_1), \nu_1(Y_2), \nu_1(Y_{12}), \nu_1(Y_{22})) = (0, -1.79, 1.79, 0, 0) \end{aligned}$$

and $\mu_1 = \frac{1}{4}$ as well as $\mu_2 = \frac{1}{9}$ solve the problem[9]

$$\begin{aligned} &\inf_{\{\nu_i(Y_s)\}} \sup_{\{c_i(Y_s)\}} \sum_{i \in \mathcal{I}} U_i(c_i) \\ &- \sum_{i \in \mathcal{I}} \mu_i \begin{pmatrix} c_i(Y_0) - \delta_i(Y_0) + \sum_s \pi(Y_s) \frac{\xi(Y_s)}{\xi(Y_0)} e^{-\nu_i(Y_s)} (c_i(Y_s) - \delta_i(Y_s)) \\ + \sum_s \pi(Y_s) \frac{\xi(Y_{s2})}{\xi(Y_s)} e^{-\nu_i(Y_s)} (c_i(Y_{s2}) - \delta_i(Y_{s2})) \\ + \theta_i(Y_0) \left(\xi(Y_0) P(Y_0) - \sum_s \pi(Y_s) \xi(Y_s) e^{-\nu_i(Y_s)} \right) \\ + \theta_i(Y_s) \left(\xi(Y_s) P(Y_s) - \xi(Y_{s2}) e^{-\nu_i(Y_s)} \right) \end{pmatrix} \end{aligned} \tag{34}$$

$$\begin{aligned} &\inf_{\{\xi(Y_s)\}} \sup_{\{c_i(Y_s)\}} \sum_{i \in \mathcal{I}} \left(\frac{1}{\mu_i} u_i(c_i(Y_0)) + \sum_s \pi(Y_s) \begin{pmatrix} \frac{1}{\mu_i e^{-\nu_i(Y_s)}} u_i(c_i(Y_s)) \\ + \frac{1}{\mu_i e^{-\nu_i(Y_s)}} u_i(c_i(Y_{s2})) \end{pmatrix} \right) \\ &- \begin{pmatrix} \sum_{i \in \mathcal{I}} \xi(Y_0) (c_i(Y_0) - \delta_i(Y_0)) + \sum_s \pi(Y_s) \sum_{i \in \mathcal{I}} \xi(Y_s) (c_i(Y_s) - \delta_i(Y_s)) \\ + \sum_s \pi(Y_s) \sum_{i \in \mathcal{I}} \xi(Y_{s2}) (c_i(Y_{s2}) - \delta_i(Y_{s2})) \end{pmatrix} \end{aligned} \tag{35}$$

where $U_i(c_i)$ is given by (28). First, one has set $1 - v_i$ equals to $e^{-\nu_i}$ in order to ensure positivity. Second, one has set $\nu_2(Y_s) \equiv 0$ equal to zero, since agent two is taken as the normative individual. Finally, $\nu_1(Y_{s2}) \equiv 0$ since period $t = 2$ correspond to a complete market situation. Notice that P is the price of the short-lived riskless bond. From (34) and (35), it is clear that Self 2 delivers the aggregate resource constraint at each date and that Self 1 accommodates the individual budget constraints as well as the kernel condition at date $t = 1$.

[9] Notice that $\overline{q}$ and $\overline{\pi}^i$ in Magill & Quinzii (1996) are obtained using

$$\begin{aligned} \overline{q}_0 &= \frac{1}{2} \times \frac{\xi(Y_1)}{\xi(Y_0)} + \frac{1}{2} \times \frac{\xi(Y_2)}{\xi(Y_0)} \\ \overline{q}_s(Y_s) &= \frac{\xi(Y_{s2})}{\xi(Y_s)} \end{aligned}$$

and

$$\overline{\pi}^i(Y_s) = \frac{1}{2} \times \xi(Y_s) \times \eta_i(Y_s).$$

5 Equilibrium & Representative Agent: Recursive Formulation

In this section, one proceeds to reformulate the equilibrium approach of section 4.4 in a dynamic framework. A recursive formulation is motivated by the high dimensionality induced by the global approach. Indeed, if one were to solve for the equilibrium in a static way, one would face a system of $(N+1)\times I+K\times I+K$ (by Walras' law) equations. For certain applications, such as continuous time approximations, this number may be very large such that one may prefer to solve for the equilibrium in a step-by-step fashion.

5.1 A Benchmark

The construction of the dynamic equivalent to the uncoordinated selves (32) and (33) is the main attempt of this section. It will be shown thereafter that this purpose requires the use of Self 1's Bellman equation associated with (32). This new aspect tends to obscure the presentation unless one has a reference point at one's disposal. In this respect, before turning to the recursive formulation of the above procedure, it is valuable to consider the benchmark method provided in Dumas & Lyasoff (2008). Their work consists in a discretization of Cuoco & He (1994) being the dynamic analogue to Cuoco & He (2001). A useful lemma provided below will serve to link Dumas & Maenhout (2003) to Dumas & Lyasoff (2008). The latter approach simply proceeds to solve (24)-(26) at each node in a backward fashion. Notice that those equations involve the discounted financial wealth as well as the discounted quoted prices: from (27) and (8), the dynamics of the latter elements are given by

$$\xi(s)F_i(s) = \sum_{s'\in S^+} \pi(s'|s)\xi(s')(1-\nu_i(s'))\left(c_i(s')-\delta_i(s')+F_i(s')\right) \qquad \forall i\in\mathcal{I} \tag{36}$$

with terminal boundary condition

$$\xi(s)F_i(s)\equiv 0 \qquad \forall s\in\mathcal{S}_T \tag{37}$$

and

$$\xi(s)P_n(s) = \sum_{s'\in S^+} \pi(s'|s)\xi(s')(1-\nu_i(s'))\left(P_n(s')+\iota_n(s')\right) \qquad \forall n\in\mathcal{N} \tag{38}$$

with terminal boundary condition

$$\xi(s)P_n(s)\equiv 0 \qquad \forall s\in\mathcal{S}_T. \tag{39}$$

As already mentioned in subsection 2.2 and made concrete by (37) and (39), the recursive aspect of the problem induces backward induction. Indeed, it is clear that every agent finish off their lives by consuming their entire wealths. Hence, it must be that their wealths reduce to nihil when reaching the terminal date T: this is reflected in (37). Similarly, assets pay off until the last date and then simply expire. Prices being discounted value of future dividends as per (38), financial claims are not worth a dime at the terminal date, i.e. (39) holds. At the same time, one is provided with initial conditions $(\xi(o),\eta_i(o))=(1,1)$ for the shadow prices. This introduces a new difficulty which was not present in the global problem since, by definition, the system is of a simultaneous nature. The main complication resides in the numerical implementation of the equilibrium which is presented in the appendix.

5.2 Two Selves and a Nash Game

One previously argued that the interaction between (32) and (33) was clearly restricted to be uncoordinated. This aspect was trivial in the global problem because, solving for the equilibrium in a simultaneous fashion, no *anticipation* was allowed. This is, however, not the case when both selves gradually move along the lattice. Indeed, moving backwards, each part of the representative individual perfectly knows the structure of the equilibrium in the future. That is, each Self perfectly anticipate the effect of a change in its control over the other Self's future optimal decisions. Therefore, in a recursive setup, a new task arises: one somehow needs to constrain each Self to make its decisions blindly. Except for being dynamic, the game in its nature remains the same: at each node $s \in \mathcal{S}$, both Selves play simultaneously a Nash game in which

- Planner 1 chooses $\{\nu_i(s') : s' \in S^+\}$ holding $\{\xi(s') : s \in S^{++}\}$ fixed
- Planner 2 chooses $\{\xi(s') : s' \in S^+\}$ holding $\{\nu_i(s') : s' \in S^{++},\ i \in \mathcal{I}\}$ fixed
- both players agree on $\{c_i(s') : s \in S^+\}$ $\forall i \in \mathcal{I}$.

Solving for Planner 2's problem, it is clear that its decision at each node $s \in \mathcal{S}$ is dictated by the accommodation of the aggregate resource constraint

$$\sum_{i \in \mathcal{I}} f_i\left(\mu_i \eta_i(s) \xi(s), s\right) = \delta(s) \tag{40}$$

such that there is no need to define the Bellman equation associated with the problem (33). Notice that Self 2's decisions are not directly impacted by $\{\nu_i(s') : s' \in S^{++},\ i \in \mathcal{I}\}$: in other words, Planner 2 has a dominant strategy. It is always a best response to Planner 1's strategy to accomodate the aggregate resource constraint. From (23) and (40), one may denote agent i's optimal consumption and Self 2's optimal action $c_i^*\left(\{\mu_i\eta_i\}, \{\delta_i\}, s\right)$ and $\xi^*\left(\{\mu_i\eta_i\}, \delta, s\right)$ respectively. It is readily observed that both equilibrium variables only depends on $\{\mu_i\eta_i\}, \{\delta_i\}$ and s such that the state variables of the game are fully characterized by the latter elements. Put differently, $\{\mu_i\eta_i\}$ and $\{\delta_i\}$ are, respectively, the endogenous and exogenous state variables. It is worth noticing that in a complete market setting, $\{\delta_i\}$ constitute a sufficient statistic for the evolution of the economy. Market incompleteness requires an additional individual specific component in order to carry a recursive computation of the equilibrium. Finally, due to its functional form, $\xi^*\left(\{\mu_i\eta_i\}, \delta, s\right)$ is homogenous of degree -1 in $\{\mu_i\eta_i\}$.

Dynamically, as both players move to the next node, nature moves to the next vertex leading to a realization of $\eta_i(s')$ $\forall s' \in S^+$ and Planner 2 accommodates the aggregate resource constraint by adjusting $\xi(s')$. Although this mechanical behavior may be anticipated by Planner 1, one makes him holding fixed when choosing his own control.

The arguments mention above are articulated in the following definition which follows Dumas & Maenhout (2003)

Definition 3. *A dynamic equilibrium of the game defined above is a set of admissible, measurable functions*

$$c_i^*(\{\eta_i\}, \{\delta_i\}, s),$$
$$\nu_i^*(\{\eta_i\}, \{\delta_i\}, s),$$

and

$$\xi^*(\{\eta_i\}, \{\delta_i\}, s)$$

such that the decisions $\{\{c_i^*\}, \{\nu_i^*\}\}$ *at node* $s \in \mathcal{S}^-$ *are optimal for Planner 1 given the process* $\{\xi^*\}$ *of the decisions of Planner 2 and given the current state variables* $\{\eta_i\}, \{\delta_i\}, s$ *and such that the decisions* $\{\{c_i^*\}, \xi^*\}$ *at node* $s \in \mathcal{S}^-$ *are optimal for Planner 2 given the process* $\{\nu_i^*\}$ *of the decisions of Planner 1 and given the current state variables* $\{\eta_i\}, \{\delta_i\}, s$.

Figure 1 adapted from Dumas & Maenhout (2003) exhibits the dynamics of the game.

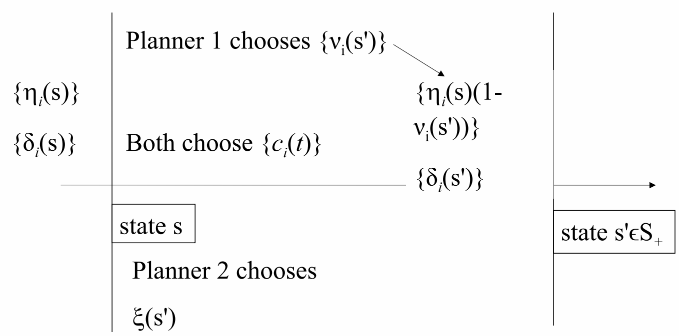

Figure 1: Dynamics of the Nash Game

The equilibrium of the game one is looking for is not a standard *Markov Perfect Equilibrium* (Fudenberg & Tirole 1994, chapter 13). Again, in a static game, the players would consider the entire control process of the other player as given. Hence, the equilibrium concept is much closer to an *Open-Loop* one (see Basar & Olsder 1999, chapter 6) except for the fact that the state space is stochastic. That is, strategies have to be Lebesgue measurable functions of the relevant state variables. More precisely, one needs the concept of *Perfect Public Equilibrium* and *Self-Generation* of Abreu, Pierce & Stacchetti (1990). Accordingly, one may prefer to regard Self 2 as a short-lived player and Self 1 as a long-run one. Indeed, since Planner 2 takes the same decision at each point in time, it may be considered as a sequence of one-shot players along the lattice[10].

5.3 Value Function

One showed that, due to the dominant aspect of its strategy, Planner 2's decision are taken independently of Planner 1's actions. As a result, Planner 2 does not anticipate Planner 1's reaction and therefore naturally acts as an uncoordinated planner. However, ξ^* depends on $\{\mu_i\eta_i\}$ such that, when Self 1 chooses $\{\nu_i\}$, it affects ξ^* through $\{\mu_i\eta_i\}$. In other words, if one were to solve for the equilibrium recursively, Self 1 would be able to anticipate Self 2's future actions when determining its own control.

[10]One may then refer to Lemma 7.1.1 in Mailath & Samuelson (2006) and establish that because self 2's strategy is public so is player 1's strategy. This implies that the probability measure is unaffected by players' moves.

In order for Self 1 to be a proper uncoordinated planner, one has to recognize two sets of variables: one that Self 1 can control $\{\mu_i\eta_i\}$ and one that it cannot $\{\mu_i\widehat{\eta}_i\}$. This is equivalent to compensate the terms involving the anticipation of Planner 2's reaction. From (32) and taking into account the two sets of controls, one writes Planner 1's *compensated* value function $\widehat{J}:\mathbb{R}_{++}^{I+1}\times\mathbb{R}_{++}^{I+1}\times\mathbb{R}_{++}^{I+1}\times\mathcal{S}\rightarrow\mathbb{R}$ as

$$
\begin{aligned}
&\widehat{J}\left(\{\mu_i\eta_i(s)\},\{\mu_i\widehat{\eta}_i(s)\},\{\delta_i\},s\right)\triangleq \\
&\sup_{\{c_i\in\mathbb{R}_{++}\}}\inf_{\{\nu_i\}}\sum_{i\in\mathcal{I}}\sum_{s'\in S^+}\pi(s'|s)u_i(c_i(s'),s') \\
&-\sum_{i\in\mathcal{I}}\sum_{s'\in S^+}\pi(s'|s)\xi^*\left(\{\mu_i\widehat{\eta}_i(s')\},\delta,s'\right)\mu_i\eta_i(s')(c_i(s')-\delta_i(s')) \\
&+\sum_{s'\in S^+}\pi(s'|s)\widehat{J}\left(\{\mu_i\eta_i(s')\},\{\mu_i\widehat{\eta}_i(s')\},\{\delta_i\},s'\right)
\end{aligned}
\tag{41}
$$

where $\{\nu_i\}$ are chosen to satisfy the pricing restriction (8). (41) is nothing but the recursive formulation of (32) where one identifies the period utilities as well as the period budget constraints aggregated over all individuals in the economy. Notice that $\{\mu_i\widehat{\eta}_i\}$ is located in ξ^* such that Planner 2's optimal decision remain actually unperturbated.

In order to obtain the first order conditions with respect to $\{\nu_i\}$, one proceeds as in the individual problem and attach the Lagrange multiplier $\mu_i\times\theta_{i,n}(s)$ to (8). The Lagrangean of that problem is given by

$$
\begin{aligned}
&\widehat{J}\left(\{\mu_i\eta_i(s)\},\{\mu_i\widehat{\eta}_i(s)\},\{\delta_i\},s\right)\triangleq \\
\sup_{\{c_i\in\mathbb{R}_{++}\}}\quad &\inf_{\{\nu_i\}}\sup_{\{\theta_i\in\Theta\}}\sum_{i\in\mathcal{I}}\sum_{s'\in S^+}\pi(s'|s)u_i(c_i(s'),s') \\
&-\sum_{i\in\mathcal{I}}\sum_{s'\in S^+}\pi(s'|s)\xi\left(\{\mu_i\widehat{\eta}_i(s')\},\delta,s'\right)\mu_i\eta_i(s')(c_i(s')-\delta_i(s')) \\
&+\sum_{s'\in S^+}\pi(s'|s)\widehat{J}\left(\{\mu_i\eta_i(s')\},\{\mu_i\widehat{\eta}_i(s')\},\{\delta_i\},s'\right) \\
&+\sum_{i\in\mathcal{I}}\theta_i^\top(s)\begin{pmatrix}\left(\sum_{s'\in S^+}\pi(s'|s)\mu_i\eta_i(s')\xi\left(\{\mu_i\widehat{\eta}_i(s')\},\delta,s'\right)\iota(s')\right)\\ -\mu_i\eta_i(s)\xi\left(\{\mu_i\widehat{\eta}_i(s)\},\delta,s\right)\mathbf{P}(s)\end{pmatrix}.
\end{aligned}
\tag{42}
$$

Using the envelope condition attached to optimal consumptions, the first order conditions of this program with respect to $\{\nu_i\}$ at each node $s\in\mathcal{S}^+$ are the $I\times m$ equations

$$
\begin{aligned}
c_i^*(s')-\frac{1}{\xi(s')}\frac{\partial}{\partial\mu_i\eta_i}\widehat{J}(s') &= \delta_i(s')+\theta_i^\top(s)(\iota(s')+\mathbf{P}(s')) \\
\forall i &\in \mathcal{I}\backslash\{I+1\} \\
\forall s' &\in S^+
\end{aligned}
\tag{43}
$$

and the $(I+1)\times N$ kernel conditions

$$\xi\left(\left\{\mu_i\widehat{\eta}_i(s)\right\},\delta,s\right)P_n(s) = \sum_{s'\in S^+(s)}\pi(s'|s)(1-\nu_i(s'))\xi\left(\left\{\mu_i\widehat{\eta}_i(s')\right\},\delta,s'\right)(\iota_n(s')+P_n(s')) \tag{44}$$
$$\forall i \in \mathcal{I}$$
$$\forall n \in \mathcal{N}.$$

One may notice that (43) looks exactly like (25) where the individual financial wealth has been replaced by $-\frac{1}{\xi}\frac{\partial}{\partial\mu_i\eta_i}\widehat{J}$. The equation (43) gets interpretated through the following lemma

Lemma 1. *The partial derivative $\frac{\partial}{\partial\mu_i\eta_i}\widehat{J}$ is equal to minus the discounted financial wealth ξF_i of investor i.*

Proof. Differentiating (41) with respect to $\mu_i\eta_i(s)$, the envelope theorem applied to optimal consumptions implies

$$\frac{\partial}{\partial\mu_i\eta_i}\widehat{J}\left(\left\{\mu_i\eta_i(s)\right\},\left\{\mu_i\widehat{\eta}_i(s)\right\},\left\{\delta_i\right\},s\right) = -\sum_{s'\in S^{++}(s)}\pi(s'|s)\xi\left(\left\{\mu_i\widehat{\eta}_i(s')\right\},\delta,s'\right)(1-\nu_i(s'))(c_i(s')-\delta_i(s')). \tag{45}$$

Q.E.D.

Hence, the main difference between the present approach and Cuoco & He (1994) is that one leads recursions over the value function of Planner 1 instead of each individual financial wealths. Accordingly, one may read (43) as the individual primal flow budget constraint. Put differently, the equilibrium of the game satisfies the optimality conditions of the individual problem (9).

5.4 Optimality Along the Equilibrium Path and Failure of the Approach

In this subsection, one first states that an equilibrium is achieved. Yet, as a second statement, its computation may not be performed along the equilibrium path only: this results into a huge loss of efficiency and discards the two central planners as a recursive equilibrium computation tool. One proceeds with the following proposition equivalently given in Dumas & Maenhout (2003):

Proposition 1. *At an equilibrium of the game, financial markets clear.*

Proof. The first order condition (43) and the kernel condition (8) imply together that individual financial wealth (27) may be rewritten as

$$\xi(s)\times F_i\left(\mu_i\eta_i(s)\xi(s),\delta_i,s\right) \equiv \sum_{s'\in S^{++}}\pi(s'|s)\xi\left(s'\right)(c_i(s')-\delta_i(s')). \tag{46}$$

From (46) and the aggregate resource constraint (40) it follows that the function $\widehat{J}$ has the property that

$$\sum_{i\in\mathcal{I}}\frac{\partial}{\partial\mu_i\eta_i}\widehat{J}\left(\left\{\mu_i\eta_i(s)\right\},\left\{\mu_i\widehat{\eta}_i(s)\right\},\left\{\delta_i\right\},s\right) \equiv 0. \tag{47}$$

Also, from (43) and using the general right-hand inverse of $\iota(s') + \mathbf{P}(s')$, agent i's portfolio choice θ_i may be expressed as

$$\theta_i^\top(s) \equiv \left(c_i(s') - \delta_i(s') - \frac{\partial}{\partial\mu_i\eta_i\xi}\widehat{J}(s')\right)(\iota(s') + \mathbf{P}(s'))^\top \left((\iota(s') + \mathbf{P}(s'))(\iota(s') + \mathbf{P}(s'))^\top\right)^{-1}. \quad (48)$$

From (47) and (48), it follows that the market clearing conditions (2) and (1) are satisfied. Q.E.D.

One showed that the equilibrium of the game achieves individual optimality and that markets clear, hence it is a market sub-equilibrium. This is, however, not a satisfactory result because the function $\widehat{J}$ involves the state space $\{\mu_i\eta_i\}$, $\{\mu_i\widehat{\eta}_i\}$, $\{\delta_i\}$ and s when the state space of the game is $\{\mu_i\eta_i\}$, $\{\delta_i\}$ and s. That is, if one desires to achieve the computation along the equilibrium path only, one uses the state space of the game. As a consequence, one would consider the equilibrium value function $J(\{\mu_i\eta_i(s)\}, \{\delta_i\}, s)$ instead of $\widehat{J}$ because, when equilibrium prevails, the two sets of state variables $\{\mu_i\eta_i\}$ and $\{\mu_i\widehat{\eta}_i\}$ have to coincide. Hence, the computation requires the use of J when (43) involves $\widehat{J}$. The only way to resolve this inconsistency is to relate $\widehat{J}$ and J. In that attempt, Dumas & Maenhout postulate the following equivalence

$$J(\{\mu_i\eta_i(s)\}, \{\delta_i\}, s) \equiv \widehat{J}(\{\mu_i\eta_i(s) \times \xi(\{\mu_i\widehat{\eta}_i(s)\}, \{\delta_i\}, s)\}), \{\delta_i\}, s) \quad (49)$$

for some function $J : \mathbb{R}_{++}^{I+1} \times \mathbb{R}_{++}^{I+1} \times \mathcal{S} \rightarrow \mathbb{R}$. Their conjecture for (49) intuitively comes from the previous observation that $\{\mu_i\widehat{\eta}_i\}$ is located in Planner 2's future decisions. However, this functional form specification of $\widehat{J}$ is not justified and does not, in general, hold. A simple counter-example is provided in the appendix. As a result, the compensation mechanism proposed in Dumas & Maenhout (2003) as

$$\frac{\partial}{\partial\mu_i\eta_i\xi}\overline{J} \equiv \frac{\frac{\partial}{\partial\mu_i\eta_i}J - \frac{\partial}{\partial\mu_i\eta_i}\xi \times \frac{\sum_i \frac{\partial}{\partial\mu_i\eta_i}J}{\sum_i \frac{\partial}{\partial\mu_i\eta_i}\xi}}{\xi} \quad (50)$$

does not work for more than a one period horizon. Put differently, the compensation is only achieved for Planner 2's decision one period ahead.

One points out that the latter result does not imply that the procedure fails for the proper equivalence. But, it is extremely doubtful that the state space of the game is sufficient to carry a recursive computation of the equilibrium. One claims but not proves that the game is strongly not subgame perfect (even more so that time inconsistency) and cannot be reproduced recursively. A game theoretical result is here needed to fully invalidate the recursive central-planning methodology.

Notice that if one were to carry a recursion on $\frac{\partial}{\partial\mu_i\eta_i}\widehat{J}$ instead of $\widehat{J}$, one would be able to compute the equilibrium in a recursive fashion. However, this approach would boil down to Cuoco & He (1994) because, as per Lemma 4, $\frac{\partial}{\partial\mu_i\eta_i}\widehat{J}$ is strictly equivalent to the financial wealth of individual i. Consequently, the two selves would be useless in determining the equilibrium allocations and, thus, theoretically meaningless. In the appendix, one provides a two periods example conceived by Professor Bernard Dumas and shows how (50) fails to compensate Planner 2's variables in period two. In the next subsection, one illustrates the magnitude of the error that one would commit using this approach.

5.5 Basak & Cuoco (1998) Limited Participation: An Illustration

One considers an economy where there are two categories of agents: Agent 1 receives an endowment stream δ which follows a geometric brownian motion. The latter is captured with a re-combining binomial tree with fixed drift and volatility as is done in Cox, Ross & Rubinstein (1979). Accordingly, for the forthcoming examples, one uses their celebrated notations. Transition probabilities are set to $\pi \equiv \frac{1}{2}$. Agent 2 receives no endowment stream but is able to consume something out of δ because he starts his life with some financial claims F on agent 1:

$$\begin{aligned} F_{2,0} &= \vartheta > 0, \\ F_{1,0} &= -\vartheta. \end{aligned}$$

Both agents are able to trade the riskless asset and the risky asset is redundant (since only agent 1 has access to it). Hence, markets are incomplete. In this example, one generalizes Basak & Cuoco (1998) to any power utility functions. The utility of both agents $i \in \{1,2\}$ at node $s \in \mathcal{S}$ is given by the isoelastic function $\beta^t \frac{c(s)^{\gamma_i}-1}{\gamma_i}$. The system of equations to solve is given and derived in the appendix. One considers $\gamma_1 = 2, \gamma_2 = 6$ and other parameters correspond to the calibration of Mehra & Prescott (1985). Figure 2 exhibits the market price of risk associated with the redundant risky asset[11] and is analoguous to Figure 2 in Basak & Cuoco (1998). The time horizon is $T = 3$ such that one should expect this difference to be much larger for remote horizon. The shape is not altered though: it should be emphasized that, unlike Basak & Cuoco, individual 2 does not have logarithmic preferences and hence the latter figure should not exactly be the same. One still observes the asymptote at 1 induced by the Inada conditions. Indeed, one recalls the formula for the market price of risk obtained by Basak & Cuoco (1998) in continuous time

$$-\frac{\frac{\partial^2}{\partial c_1^2} u_1(c_1(t),t)}{\frac{\partial}{\partial c_1} u_1(c_1(t),t)} \sigma_\delta(t).$$

Notice that one considers CRRA utility functions and hence $-\frac{\frac{\partial^2}{\partial c_1^2} u_1(c_1(t),t)}{\frac{\partial}{\partial c_1} u_1(c_1(t),t)}$ is not a constant. Furthermore, when $\omega \to 1$, one simultaneously deduces that $c_2 \to \infty$. Hence, by Inada conditions, $\frac{\partial}{\partial c_1} u_1(c_1(t),t) \to 0$ and thus, by concavity of the isoelastic utility function, $-\frac{\frac{\partial^2}{\partial c_1^2} u_1(c_1(t),t)}{\frac{\partial}{\partial c_1} u_1(c_1(t),t)} \sigma_\delta(t) \to \infty$. It is also reminded that the target for the market price of risk is 0.4 and that it may be difficult to reach it in this setting. However, note that market incompleteness considerably improves the market price of risk's fit to the data.

6 Conclusion

One showed that a central planning procedure involving two selves may not be achieved within a recursive incomplete markets setting. Such a procedure requires a much larger state space than the one described in the Nash game. Hence, two ways around this approach include performing the equilibrium over the entire state space or carrying recursion over the derivative of Planner 1's value

[11] The derivation of the market price of risk is provided in the appendix.

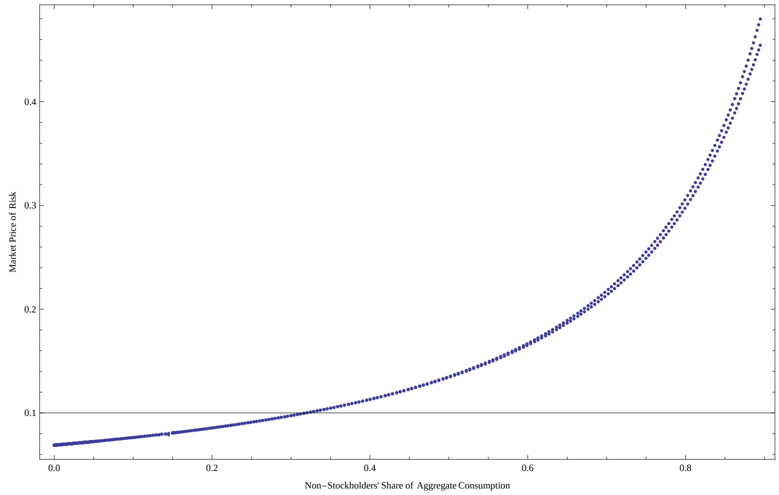

Figure 2: Market Price of Risk in the Basak & Cuoco (1998) Economy

function. The former results in a huge loss of efficiency and the latter boils down to Cuoco & He (1994) imperfectly centralized methodology. A proper central planning approach seems to require as many central planners as there are states in the economy. However, this raises a dimensionality issue for remote horizon applications. It is of theoretical importance for future research to enquire whether a complete central planning methodology may be manufactured at slight efficiency costs.

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7 Appendix

This appendix mainly contains notes provided by Professor Dumas: Mathematica 6.0 codes were adapted from those that Professor Dumas used in Dumas & Lyasoff (2008). The market price of risk computation and the two periods examples are directly taken from Professor Dumas notes.

7.1 Numerical Methods

If (50) were a proper compensating mechanism, then (43) and (44) would involve at each node the functions J (in Dumas & Laysoff (2008), it involves (36)) and ξP. These functions need to be interpolated. In order to make this task easier, one uses the homogeneity of degree 0 and -1 in η_i of J and ξP respectively. Accordingly, one interpolates

$$j\left(\{\omega_i\}_{i\in\mathcal{I}\backslash I+1},\{\delta_i\},s\right)\triangleq J\left(\{\mu_i\eta_i(s)\},\{\delta_i\},s\right) \tag{51}$$

and

$$(\xi p)_n\left(\{\omega_i\}_{i\in\mathcal{I}\backslash I+1},\{\delta_i\},s\right)\triangleq\left(\sum_{i\in\mathcal{I}}\eta_i\right)\times(\xi P)_n\left[\{\mu_i\eta_i(s)\},\{\delta_i\},s\right] \tag{52}$$

with $\omega_i\triangleq\frac{\mu_i\eta_i}{\sum_{i\in\mathcal{I}}\eta_i}$ $\forall i\in\mathcal{I}\backslash I+1, s\in\mathcal{S}$. In the present work, this is achieved through the Mathematica 6.0 `Interpolation` command which creates an `InterpolatingFunctions` that uses Lagrange local polynomials of order 3 to relate points. In the case of Dumas & Maenhout (2003), one should consider higher order polynomials due to the derivative of the function J involved in the equations system. The interpolation is achieved over a `Domain` and one makes sure that no extrapolation is used by attributing the value `Indeterminate` to any point falling out of the `Domain`.

The `Domain` used for $\{\omega_i\}_{i\in\mathcal{I}\backslash I+1}$ is trivially given by $[0,1]$. Due to the requirement that utility functions should satisfy the Inada conditions, j and $(\xi p)_n$ have vertical asymptotes at the boundaries of the $\omega-$domain. Hence, one should be very careful to consider $(0,1)$ as the actual domain for practical purposes. One may still get arbitrarily close to $\omega=0$ and $\omega=1$ by choosing ε small enough such that, at the boundaries, one has $\omega=\varepsilon$ and $\omega=1-\varepsilon$. Also, still due to the asymptotic behaviour of the interpolating functions close to the closure of the domain, the $\omega-$grid may be difficult to construct. One takes care of this problem by turning to another one: Following Hazaveh & al. (2003), one solves by homotopy the system of equations for all nodes at time $t=T-1$ for values of $\{\omega_i\}_{i\in\mathcal{I}\backslash I+1}$ in a range. Notice that this date is chosen because at that time, the system of equations does not involve the functions j and $(\xi p)_n$ (those functions are equal to zero). One solves the system numerically at some benchmark point $\{\omega_i^*\}_{i\in\mathcal{I}\backslash I+1}$. One then differentiates once the latter system with respect to $\{\omega_i\}_{i\in\mathcal{I}\backslash I+1}$ and consider the resulting equations as a system of ordinary first-degree differential equations to be solved numerically with benchmark point as initial condition. Using Mathematica 6.0, this task is achieved through `NDSolve`. This command generates an interpolating function over an automatically chosen grid that one extracts and uses for the entire recursive calculation. Hence, proceeding with the grid and interpolating j and $(\xi p)_n$, one deals with the forward-backward aspect of the problem.

The Figure below exhibits a typical grid that is obtained using the procedure just described. As one may observe, points are accumulated near the boundaries of the $\omega-$domain where the functions may be harder to interpolate due to the asymptotes.

7.2 Basak & Cuoco (1998) Limited Participation

The system of equations (40), (43) and (44) particularizes to

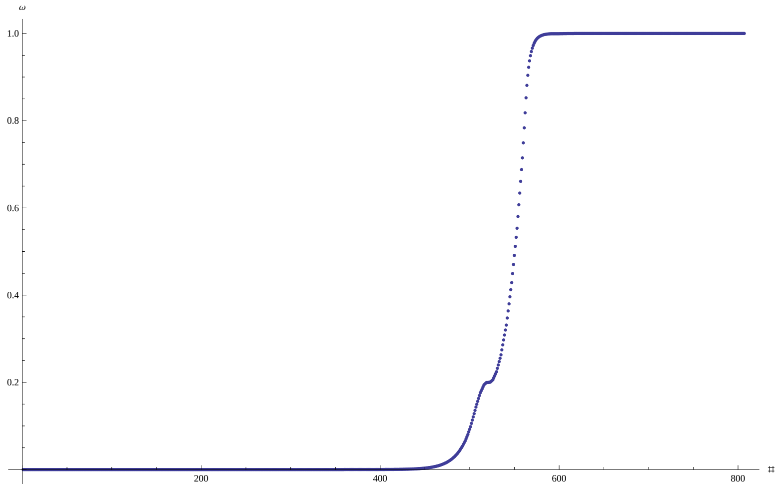

Figure 3: Representative Grid Used for ω

$$\left(\frac{\mu_1 \xi_{t+1,u} \eta_{1,t}(1-\nu_{1,t+1,u})}{\beta^{t+1}}\right)^{\frac{1}{\gamma_1-1}} - \frac{J_{1,t+1,u} - \frac{\sum_{k=1,2} J_{k,t+1,u}}{\sum_{k=1,2} \xi_{k,t+1,u}} \times \xi_{1,t+1,u}}{\xi_{t+1,u}} = \delta_{t+1,u} + \alpha_{1,t} \quad (53)$$

$$\left(\frac{\mu_2 \xi_{t+1,u} \eta_{2,t}(1-\nu_{2,t+1,u})}{\beta^{t+1}}\right)^{\frac{1}{\gamma_2-1}} - \frac{J_{2,t+1,u} - \frac{\sum_{k=1,2} J_{k,t+1,u}}{\sum_{k=1,2} \xi_{k,t+1,u}} \times \xi_{2,t+1,u}}{\xi_{t+1,u}} = \alpha_{2,t} \quad (54)$$

$$\left(\frac{\mu_1 \xi_{t+1,d} \eta_{1,t}(1-\nu_{1,t+1,d})}{\beta^{t+1}}\right)^{\frac{1}{\gamma_1-1}} - \frac{J_{1,t+1,d} - \frac{\sum_{k=1,2} J_{k,t+1,d}}{\sum_{k=1,2} \xi_{k,t+1,d}} \times \xi_{1,t+1,d}}{\xi_{t+1,d}} = \delta_{t+1,d} + \alpha_{1,t} \quad (55)$$

$$\left(\frac{\mu_2 \xi_{t+1,d} \eta_{2,t}(1-\nu_{2,t+1,d})}{\beta^{t+1}}\right)^{\frac{1}{\gamma_2-1}} - \frac{J_{2,t+1,d} - \frac{\sum_{k=1,2} J_{k,t+1,d}}{\sum_{k=1,2} \xi_{k,t+1,d}} \times \xi_{2,t+1,d}}{\xi_{t+1,d}} = \alpha_{2,t} \quad (56)$$

for (43)

$$\frac{1}{2}\xi_{t+1,u} + \frac{1}{2}\xi_{t+1,d} = \frac{1}{2}\xi_{t+1,u} \times (1-\nu_{1,t+1,u}) + \frac{1}{2}\xi_{t+1,d} \times (1-\nu_{1,t+1,d}) \quad (57)$$

for (44) and

$$\left(\frac{\mu_1 \xi_{t+1,d} \eta_{1,t}(1-\nu_{1,t+1,d})}{\beta^{t+1}}\right)^{\frac{1}{\gamma_1-1}} + \left(\frac{\mu_2 \xi_{t+1,d} \eta_{2,t}(1-\nu_{2,t+1,d})}{\beta^{t+1}}\right)^{\frac{1}{\gamma_2-1}} = \delta_{t+1,d} \quad (58)$$

$$\left(\frac{\mu_1 \xi_{t+1,u} \eta_{1,t}(1-\nu_{1,t+1,u})}{\beta^{t+1}}\right)^{\frac{1}{\gamma_1-1}} + \left(\frac{\mu_2 \xi_{t+1,u} \eta_{2,t}(1-\nu_{2,t+1,u})}{\beta^{t+1}}\right)^{\frac{1}{\gamma_2-1}} = \delta_{t+1,u} \quad (59)$$

for (40). Along with the recursive formulation of the equilibrium value function J is given by

$$J(\mu_1\eta^*_{1,t},\mu_2\eta^*_{2,t},\delta_t,t)\equiv \tag{60}$$
$$\frac{1}{2}\left(\begin{array}{c}\left(\frac{1}{\gamma_1}-1\right)\beta^{\frac{t+1}{1-\gamma_1}}\left(\xi^*_{t+1,u}\eta^*_{1,t}(1-\nu^*_{1,t+1,u})\right)^{\frac{\gamma_1}{\gamma_1-1}}\\ +\left(\frac{1}{\gamma_2}-1\right)\beta^{\frac{t+1}{1-\gamma_2}}\left(\xi^*_{t+1,u}\eta^*_{2,t}\right)^{\frac{\gamma_2}{\gamma_2-1}}\\ +\xi^*_{t+1,u}\eta^*_{1,t}(1-\nu^*_{1,t+1,u})\delta_{t+1,u}+J(\mu_1\eta^*_{1,t}(1-\nu^*_{1,t+1,u}),\mu_2\eta^*_{2,t},\delta_{t+1},t+1)\end{array}\right)$$
$$+\frac{1}{2}\left(\begin{array}{c}\left(\frac{1}{\gamma_1}-1\right)\beta^{\frac{t+1}{1-\gamma_1}}\left(\xi^*_{t+1,d}\eta^*_{1,t}(1-\nu^*_{1,t+1,d})\right)^{\frac{\gamma_1}{\gamma_1-1}}\\ +\left(\frac{1}{\gamma_2}-1\right)\beta^{\frac{t+1}{1-\gamma_2}}\left(\xi^*_{t+1,d}\eta^*_{2,t}\right)^{\frac{\gamma_2}{\gamma_2-1}}\\ +\xi^*_{t+1,d}\eta^*_{1,t}(1-\nu^*_{1,t+1,d})\delta_{t+1,d}+J(\mu_1\eta^*_{1,t}(1-\nu^*_{1,t+1,d}),\mu_2\eta^*_{2,t},\delta_{t+1},t+1)\end{array}\right).$$

Notice that the error made when using this methodology is induced by the compensating terms in (53)-(56): one simply replaces those elements by the individual undiscounted financial wealths to recover Dumas & Lyasoff (2008). The system of seven equations (53)-(59) has six unknowns[12] one of which is redundant. Indeed, by adding (53) and (54), using (59) as well as (47), one gets an identity: $0 = 0$. This allows to get rid of one of the four first equations (53)-(56) and get a system of six equations in six unknowns. One is able to get rid of (53) which would only serve to determine α_1. Furthermore, one is able to completely rule out the portfolio choice by matching both remaining first order conditions (55) and (56). Hence, one is left with a system of four equations in four unknowns:

$$\left(\frac{\mu_2\xi_{t+1,u}\eta_{2,t}}{\beta^{t+1}}\right)^{\frac{1}{\gamma_2-1}}-\frac{J_{2,t+1,u}-\frac{\sum_{k=1,2}J_{k,t+1,u}}{\sum_{k=1,2}\xi_{k,t+1,u}}\times\xi_{2,t+1,u}}{\xi_{t+1,u}}- \tag{61}$$
$$\left(\frac{\mu_2\xi_{t+1,d}\eta_{2,t}}{\beta^{t+1}}\right)^{\frac{1}{\gamma_2-1}}+\frac{J_{2,t+1,d}-\frac{\sum_{k=1,2}J_{k,t+1,d}}{\sum_{k=1,2}\xi_{k,t+1,d}}\times\xi_{2,t+1,d}}{\xi_{t+1,d}}=0$$

$$\frac{1}{2}\xi_{t+1,u}+\frac{1}{2}\xi_{t+1,d}=\frac{1}{2}\xi_{t+1,u}\times(1-\nu_{1,t+1,u})+\frac{1}{2}\xi_{t+1,d}\times(1-\nu_{1,t+1,d}) \tag{62}$$

$$\left(\frac{\mu_1\xi_{t+1,d}\eta_{1,t}(1-\nu_{1,t+1,d})}{\beta^{t+1}}\right)^{\frac{1}{\gamma_1-1}}+\left(\frac{\mu_2\xi_{t+1,d}\eta_{2,t}}{\beta^{t+1}}\right)^{\frac{1}{\gamma_2-1}}=\delta_{t+1,d} \tag{63}$$

$$\left(\frac{\mu_1\xi_{t+1,u}\eta_{1,t}(1-\nu_{1,t+1,u})}{\beta^{t+1}}\right)^{\frac{1}{\gamma_1-1}}+\left(\frac{\mu_2\xi_{t+1,u}\eta_{2,t})}{\beta^{t+1}}\right)^{\frac{1}{\gamma_2-1}}=\delta_{t+1,u}\ . \tag{64}$$

Notice that in Dumas & Lyasoff, one used agent 2's discounted financial wealth

$$\xi_tF_{2,t}=\frac{1}{2}\left[\xi_{t+1,u}\times\left(\frac{\mu_2\xi_{t+1,u}\times\eta_{2,t}}{\rho^{t+1}}\right)^{\frac{1}{\gamma_2-1}}+\xi_{t+1,u}F_{2,t+1,u}\right] \tag{65}$$
$$+\frac{1}{2}\left[\xi_{t+1,d}\times\left(\frac{\mu_2\xi_{t+1,d}\times\eta_{1,t}}{\rho^{t+1}}\right)^{\frac{1}{\gamma_2-1}}+\xi_{t+1,d}F_{2,t+1,d}\right].$$

[12] $\{\nu_{2,u},\nu_{2,d}\}$ dropped out of the system since the second agent normalizes the system and accordingly, one set $\nu_2\equiv 0$.

The intuition for (61) is that agent 2 tries to smooth his consumption across states. It also reflects that the only asset available is the riskless one. This is confirmed in the kernel condition (62) where the price drops out because the short-lived bond is worth nothing next period and where the dividend is just one. Finally, in (65) and (60) one checks that agent 2 receives no endowment (the total aggregate output belongs to agent 1).

7.3 Market Price of Risk

One considers the dynamics of the market price of the risky claim

$$\begin{aligned} \xi_t P_t &= \frac{1}{2}(1-\nu_{1,t,u}) \times [\xi_{t+1,u} \times \delta_{t+1,u} + \xi_{t+1,u} P_{t+1,u}] \\ &+ \frac{1}{2}(1-\nu_{1,t,d}) \times [\xi_{t+1,d} \times \delta_{t+1,d} + \xi_{t+1,d} P_{t+1,d}]\,; t = 0,1,...T-1 \\ \xi_T P_T &= 0. \end{aligned} \tag{66}$$

Dividing the latter expression by $\xi_t P_t$

$$\begin{aligned} 1 &= \frac{1}{2}(1-\nu_{1,t,u}) \times \frac{\xi_{t+1,u}}{\xi_t}\frac{\delta_{t+1,u}+P_{t+1,u}}{P_t} \\ &+ \frac{1}{2}(1-\nu_{1,t,d}) \times \frac{\xi_{t+1,d}}{\xi_t}\frac{\delta_{t+1,d}+P_{t+1,d}}{P_t}. \end{aligned} \tag{67}$$

Adding and substituting $\left(\frac{1}{2}\frac{\xi_{t+1,u}}{\xi_t}+\frac{1}{2}\frac{\xi_{t+1,d}}{\xi_t}\right) \times \left(\frac{1}{2}\frac{\delta_{t+1,u}+P_{t+1,u}}{P_t}+\frac{1}{2}\frac{\delta_{t+1,d}+P_{t+1,d}}{P_t}\right)$

$$\begin{aligned} 1 &= \frac{1}{2}\left[(1-\nu_{1,t,u}) \times \frac{\xi_{t+1,u}}{\xi_t} - \left(\frac{1}{2}\frac{\xi_{t+1,u}}{\xi_t}+\frac{1}{2}\frac{\xi_{t+1,d}}{\xi_t}\right)\right]\frac{\delta_{t+1,u}+P_{t+1,u}}{P_t} \\ &+ \frac{1}{2}\left[(1-\nu_{1,t,d}) \times \frac{\xi_{t+1,d}}{\xi_t} - \left(\frac{1}{2}\frac{\xi_{t+1,u}}{\xi_t}+\frac{1}{2}\frac{\xi_{t+1,d}}{\xi_t}\right)\right]\frac{\delta_{t+1,d}+P_{t+1,d}}{P_t} \\ &+ \left(\frac{1}{2}\frac{\xi_{t+1,u}}{\xi_t}+\frac{1}{2}\frac{\xi_{t+1,d}}{\xi_t}\right) \times \left(\frac{1}{2}\frac{\delta_{t+1,u}+P_{t+1,u}}{P_t}+\frac{1}{2}\frac{\delta_{t+1,d}+P_{t+1,d}}{P_t}\right). \end{aligned} \tag{68}$$

Rearranging,

$$\begin{aligned} &\left(\frac{1}{2}\frac{\xi_{t+1,u}}{\xi_t}+\frac{1}{2}\frac{\xi_{t+1,d}}{\xi_t}\right) \times \left(\frac{1}{2}\frac{\delta_{t+1,u}+P_{t+1,u}}{P_t}+\frac{1}{2}\frac{\delta_{t+1,d}+P_{t+1,d}}{P_t}\right) - 1 \\ &= -\frac{1}{2}\left[(1-\nu_{1,t,u}) \times \frac{\xi_{t+1,u}}{\xi_t} - \left(\frac{1}{2}\frac{\xi_{t+1,u}}{\xi_t}+\frac{1}{2}\frac{\xi_{t+1,d}}{\xi_t}\right)\right]\frac{\delta_{t+1,u}+P_{t+1,u}}{P_t} \\ &-\frac{1}{2}\left[(1-\nu_{1,t,d}) \times \frac{\xi_{t+1,d}}{\xi_t} - \left(\frac{1}{2}\frac{\xi_{t+1,u}}{\xi_t}+\frac{1}{2}\frac{\xi_{t+1,d}}{\xi_t}\right)\right]\frac{\delta_{t+1,d}+P_{t+1,d}}{P_t}. \end{aligned} \tag{69}$$

Finally, divide by the standard deviation of the return and get

$$\frac{\left(\frac{1}{2}\frac{\xi_{t+1,u}}{\xi_t}+\frac{1}{2}\frac{\xi_{t+1,d}}{\xi_t}\right)\times\left(\frac{1}{2}\frac{\delta_{t+1,u}+P_{t+1,u}}{P_t}+\frac{1}{2}\frac{\delta_{t+1,d}+P_{t+1,d}}{P_t}\right)-1}{\frac{1}{2}\left(\frac{\delta_{t+1,u}+P_{t+1,u}}{P_t}-\frac{\delta_{t+1,d}+P_{t+1,d}}{P_t}\right)} \tag{70}$$

$$= -\left[(1-\nu_{1,t,u})\times\frac{\xi_{t+1,u}}{\xi_t}-\left(\frac{1}{2}\frac{\xi_{t+1,u}}{\xi_t}+\frac{1}{2}\frac{\xi_{t+1,d}}{\xi_t}\right)\right]$$

$$= (1-\nu_{1,t,d})\times\frac{\xi_{t+1,d}}{\xi_t}-\left(\frac{1}{2}\frac{\xi_{t+1,u}}{\xi_t}+\frac{1}{2}\frac{\xi_{t+1,d}}{\xi_t}\right).$$

7.4 Functional Form Misspecification: A Counter-Example

In the perspective of illustrating the functional form misspecification of the function $\widehat{J}$, one considers a very stylized example that allows to derive the value function J in closed-form. More precisely, one particularizes Basak & Cuoco (1998) to fully logarithmic preferences with equal impatience rate ρ and infinite horizon. It is best for this example to operate computations in a continuous time framework: the aggregate resource constraint (60) and (65) in Dumas & Maenhout (2003) reduce to

$$\left(\xi^{**}(\omega,\delta)\,\omega\right)^{-1}+\left(\xi^{**}(\omega,\delta)\,(1-\omega)\right)^{-1}=\delta \tag{71}$$

and

$$\nu_1\left(\xi^{**}(\omega,\delta)\,\omega\right)^{-1}=\overline{\sigma}\delta \tag{72}$$

respectively. On the other hand, the PDE (67) particularizes to

$$\begin{aligned} 0 \;=\;& -\log\xi^{**}\omega-\log\xi^{**}(1-\omega)-2+\xi^{**}\omega\delta-\rho I \\ &+I_\delta\overline{\mu}\delta+\frac{1}{2}I_{\delta\delta}\overline{\sigma}^2\delta^2-I_\omega\omega^2(1-\omega)\nu_1^2 \\ &+\frac{1}{2}I_{\omega\omega}((1-\omega)\omega\nu_1)^2-I_{\omega\delta}(1-\omega)\omega\nu_1\overline{\sigma}\delta. \end{aligned} \tag{73}$$

From (71) and (72), one deduces

$$\xi^{**}(\omega,\delta)=(\delta\omega(1-\omega))^{-1} \tag{74}$$

as well as

$$\nu_1=\frac{\overline{\sigma}}{1-\omega} \tag{75}$$

which are substituted in (73) to obtain

$$\begin{aligned} 0 \;=\;& 2\log\delta+\log(1-\omega)+\log\omega-2+\frac{1}{1-\omega}-\rho I \\ &+I_\delta\overline{\mu}\delta+\frac{1}{2}I_{\delta\delta}\overline{\sigma}^2\delta^2-I_\omega\frac{\overline{\sigma}^2\omega^2}{1-\omega} \\ &+\frac{1}{2}I_{\omega\omega}\omega^2\overline{\sigma}^2-I_{\omega\delta}\omega\overline{\sigma}^2\delta. \end{aligned} \tag{76}$$

Then, one makes the conjecture for the value function I

$$I(\omega,\delta)\equiv a(\omega)+\frac{2}{\rho}\log\delta \tag{77}$$

which one injects back into (76) and collects the ODE

$$\begin{aligned}0 &= \log(1-\omega)+\log\omega-2+\frac{1}{1-\omega}-\rho a(\omega) \\ &+\frac{2}{\rho}\overline{\mu}-\frac{1}{2\rho}\overline{\sigma}^2-a'(\omega)\frac{\overline{\sigma}^2\omega^2}{1-\omega}+\frac{1}{2}a''(\omega)\omega^2\overline{\sigma}^2.\end{aligned} \tag{78}$$

The solution for the function a is of the form

$$a(\omega)=C_1\omega^{r_1}+C_2\omega^{r_2}+P(\omega) \tag{79}$$

where the roots are

$$r_1\equiv\frac{\overline{\sigma}+\sqrt{8\rho+\overline{\sigma}^2}}{2\overline{\sigma}}>0, r_2\equiv\frac{\overline{\sigma}-\sqrt{8\rho+\overline{\sigma}^2}}{2\overline{\sigma}}<0. \tag{80}$$

Since $r_2<0$, one sets $C_2=0$.Then, J is recovered using $J(\mu_1\eta_1,\mu_2\eta_2,\delta)=a(\frac{\mu_1\eta_1}{\mu_1\eta_1+\mu_2\eta_2})+\frac{2}{\rho}\log\delta$ along with the function a given by

$$a(\omega)\equiv\frac{1}{\omega-1}\left\{\begin{array}{c}\frac{\overline{\sigma}\omega}{\rho\sqrt{8\rho+\overline{\sigma}^2}}A-\frac{1}{2\rho^2(\rho-\overline{\sigma}^2)}\left(4\overline{\mu}\rho-2\rho^2-4\overline{\mu}\rho\overline{\sigma}^2+2\overline{\sigma}^2-4\left(\overline{\mu}\rho-\rho^2-\overline{\mu}\overline{\sigma}^2+\rho\overline{\sigma}^2\right)\omega\right) \\ +\frac{2\overline{\sigma}\rho(\rho-\overline{\sigma}^2)}{\sqrt{8\rho+\overline{\sigma}^2}}B+\frac{\omega^2\rho\overline{\sigma}\left(\overline{\sigma}\sqrt{8\rho+\overline{\sigma}^2}+\rho+\overline{\sigma}^2\right)}{\overline{\sigma}\sqrt{8\rho+\overline{\sigma}^2}}D+\frac{\omega^2\rho\overline{\sigma}\left(\overline{\sigma}\left(\sqrt{8\rho+\overline{\sigma}^2}-\overline{\sigma}\right)-2\rho\right)}{\sqrt{8\rho+\overline{\sigma}^2}}C \\ +2C_1\rho^2\left(\rho-\overline{\sigma}^2\right)\omega^{r_1}+2\rho\left(\rho-\overline{\sigma}^2+\omega\left(\overline{\sigma}^2-\rho\right)\right)\left(\log(1-\omega)+\log\omega\right)\end{array}\right\} \tag{81}$$

with

$$\begin{aligned}A &\equiv {}_2F_1\left[r_2,1,1+r_2,\omega\right],\ B\equiv{}_2F_1\left[r_1,1,1+r_1,\omega\right], \\ C &\equiv {}_2F_1\left[1+r_1,1,2+r_1,\omega\right],\ D\equiv{}_2F_1\left[1+r_2,1,2+r_2,\omega\right]\end{aligned} \tag{82}$$

where ${}_2F_1$ stands for the hypergeometric function of order 2. One left the constant of integration C_1 undertermined because no obvious condition was available. Nevertheless, it is quite clear from (81) that the value function is not going to have the functional form postulated in (49) such that the compensation mechanism (50) does not work.

7.5 Failure of the Compensation Mechanism: A 2 Period Example

This example has been prepared by Professor Dumas and shows that the compensation mechanism in Dumas & Maenhout (2003) does not work two periods ahead. One considers an economy with two periods and two agents only and proceeds to show that the compensation mechanism is unable to rule out period two's anticipating terms. Consider the period one utility of Planner 1

$$U\left(c_1,c_2,\xi\eta_1,\xi\eta_2,u,t\right)=\beta^t u_1\left(c_1\right)+\beta^t u_2\left(c_2\right)-\mu_1\xi\eta_1\left(c_1-\delta\right)-\mu_2\xi\eta_2 c_2 \tag{83}$$

where optimal consumptions are given by

$$c_{1,t} = u_1'^{-1}\left(\frac{\mu_1 \xi_t \eta_{1,t}}{\beta^t}\right) \tag{84}$$

and

$$c_{2,t} = u_2'^{-1}\left(\frac{\mu_2 \xi_t \eta_{2,t}}{\beta^t}\right). \tag{85}$$

Let X be the expected value at time t of the next period's value of the aggregator. It is a function, to be made explicit, of this period's η_1 and η_2 and δ.

Throughout, as a normalization, one sets $\nu_{2,t}$ equal to zero.

Consider consumptions at time $t = T-1$:

$$\begin{aligned} c_{1,T,u} &= u_1'^{-1}\left(\frac{\mu_1 \xi_{T,u} \eta_{1,T-1}\left(1-\nu_{1,T,u}\right)}{\beta^T}\right); c_{1,T,d} = u_1'^{-1}\left(\frac{\mu_1 \xi_{T,d} \eta_{1,T-1}\left(1-\nu_{1,T,d}\right)}{\beta^T}\right) \\ c_{2,T,u} &= u_2'^{-1}\left(\frac{\mu_2 \xi_{T,u} \eta_{2,T-1}}{\beta^T}\right); c_{2,T,d} = u_2'^{-1}\left(\frac{\mu_2 \xi_{T,d} \eta_{2,T-1}}{\beta^T}\right) \end{aligned} \tag{86}$$

along with the following objective function

$$\overline{X}_{T-1} = \frac{1}{2} X_{T,u} + \frac{1}{2} X_{T,d} \tag{87}$$

where

$$\begin{aligned} X_{T,u} &= \beta^T u_1\left(c_{1,T,u}\right) + \beta^T u_2\left(c_{2,T,u}\right) \\ &\quad -\mu_1 \xi_{T,u} \eta_{1,T-1}\left(1-\nu_{1,T,u}\right)\left(c_{1,T,u}-\delta_{T,u}\right) - \mu_2 \xi_{T,u} \eta_{2,T-1} c_{2,T,u} \\ X_{T,d} &= \beta^T u_1\left(c_{1,T,d}\right) + \beta^T u_2\left(c_{2,T,d}\right) \\ &\quad -\mu_1 \xi_{T,d} \eta_{1,T-1}\left(1-\nu_{1,T,d}\right)\left(c_{1,T,d}-\delta_{T,d}\right) - \mu_2 \xi_{T,d} \eta_{2,T-1} c_{2,T,d} \end{aligned} \tag{88}$$

being minimized subject to the pricing/kernel restriction

$$\frac{1}{2} \xi_{T,u} \eta_{1,T-1}\left(1-\nu_{1,T,u}\right) + \frac{1}{2} \xi_{T,d} \eta_{1,T-1}\left(1-\nu_{1,T,d}\right) = \frac{1}{2} \xi_{T,u} \eta_{1,T-1} + \frac{1}{2} \xi_{T,d} \eta_{1,T-1}. \tag{89}$$

Let one call $\alpha_{1,T-1}\mu_1$ the Lagrange multiplier of the restriction and form the Lagrangean:

$$\frac{1}{2} L_{T,u} + \frac{1}{2} L_{T,d} \tag{90}$$

where

$$\begin{aligned} L_{T,u} &= \beta^T u_1\left(c_{1,T,u}\right) + \beta^T u_2\left(c_{2,T,u}\right) - \mu_1 \xi_{T,u} \eta_{1,T-1}\left(1-\nu_{1,T,u}\right)\left(c_{1,T,u}-\delta_{T,u}\right) \\ &\quad -\mu_2 \xi_{T,u} \eta_{2,T-1} c_{2,T,u} + \alpha_{1,T-1}\mu_1\left[\xi_{T,u} \eta_{1,T-1} - \xi_{T,u} \eta_{1,T-1}\left(1-\nu_{1,T,u}\right)\right] \\ L_{T,d} &= \beta^T u_1\left(c_{1,T,d}\right) + \beta^T u_2\left(c_{2,T,d}\right) - \mu_1 \xi_{T,d} \eta_{1,T-1}\left(1-\nu_{1,T,d}\right)\left(c_{1,T,d}-\delta_{T,d}\right) \\ &\quad -\mu_2 \xi_{T,d} \eta_{2,T-1} c_{2,T,d} + \alpha_{1,T-1}\mu_1\left[\xi_{T,d} \eta_{1,T-1} - \xi_{T,d} \eta_{1,T-1}\left(1-\nu_{1,T,d}\right)\right] \end{aligned} \tag{91}$$

Optimizing w.r.t. $\nu_{1,T,u}$ and $\nu_{1,T,d}$, keeping the ξs fixed and using the envelope condition for consumptions, one obtains the two first conditions

$$\begin{aligned}
\left.\frac{\partial\left(\frac{1}{2}L_{T,u}+\frac{1}{2}L_{T,d}\right)}{\partial\nu_{1,T,u}}\right|_{\xi_{T,u},\xi_{T,d}} &= \frac{1}{2}\left.\frac{\partial L_{T,u}}{\partial\nu_{1,T,u}}\right|_{\xi_{T,u}} && (92)\\
&= \frac{1}{2}\left[\mu_1\eta_{1,T-1}\left(\xi_{T,u}\left(c_{1,T,u}-\delta_{T,u}\right)+\alpha_{1,T-1}\xi_{T,u}\right)\right]\\
&= 0 && (93)
\end{aligned}$$

$$\begin{aligned}
\left.\frac{\partial\left(\frac{1}{2}L_{T,u}+\frac{1}{2}L_{T,d}\right)}{\partial\nu_{1,T,d}}\right|_{\xi_{T,u},\xi_{T,d}} &= \frac{1}{2}\left.\frac{\partial L_{T,d}}{\partial\nu_{1,T,d}}\right|_{\xi_{T,d}} && (94)\\
&= \frac{1}{2}\left[\mu_1\eta_{1,T-1}\left(\xi_{T,d}\left(c_{1,T,d}-\delta_{T,d}\right)+\alpha_{1,T-1}\xi_{T,d}\right)\right]\\
&= 0 && (95)
\end{aligned}$$

which boils down to

$$c_{2,T,u}=c_{2,T,d} \qquad (96)$$

and, therefore

$$\begin{aligned}
& u_1'^{-1}\left(\frac{\mu_1\xi_{T,u}\eta_{1,T-1}\left(1-\nu_{1,T,u}\right)}{\beta^T}\right)-u_1'^{-1}\left(\frac{\mu_1\xi_{T,d}\eta_{1,T-1}\left(1+\nu_{1,T,d}\right)}{\beta^T}\right) && (97)\\
=\ & \delta_{T,u}-\delta_{T,d}.
\end{aligned}$$

In the isoelastic case, this particularizes to

$$\begin{aligned}
& \left(\frac{\mu_1\xi_{T,u}\eta_{1,T-1}\left(1-\nu_{1,T,u}\right)}{\beta^T}\right)^{\frac{1}{\gamma_1-1}}-\left(\frac{\mu_1\xi_{T,d}\eta_{1,T-1}\left(1-\nu_{1,T,d}\right)}{\beta^T}\right)^{\frac{1}{\gamma_1-1}} && (98)\\
=\ & \delta_{T,u}-\delta_{T,d}
\end{aligned}$$

Here are Planner 2's equations, which will hold in equilibrium (including the fact that $\nu_{2,T-1}=0$):

$$\begin{aligned}
u_1'^{-1}\left(\frac{\mu_1\xi_{T,u}\eta_{1,T-1}\left(1-\nu_{1,T,u}\right)}{\beta^T}\right)+u_2'^{-1}\left(\frac{\mu_2\xi_{T,u}\eta_{2,T-1}}{\beta^T}\right) &= \delta_{T,u} && (99)\\
u_1'^{-1}\left(\frac{\mu_1\xi_{T,d}\eta_{1,T-1}\left(1+\nu_{1,T,d}\right)}{\beta^T}\right)+u_2'^{-1}\left(\frac{\mu_2\xi_{T,d}\eta_{2,T-1}}{\beta^T}\right) &= \delta_{T,d} && (100)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\mu_1\xi_{T,u}\eta_{1,T-1}\times\left(1-\nu_{1,T,u}\right)}{\beta^T}\right)^{\frac{1}{\gamma_1-1}}+\left(\frac{\mu_2\xi_{T,u}\eta_{2,T-1}}{\beta^T}\right)^{\frac{1}{\gamma_2-1}} &= \delta_{T,u} && (101)\\
\left(\frac{\mu_1\xi_{T,d}\eta_{1,T-1}\times\left(1-\nu_{1,T,d}\right)}{\beta^T}\right)^{\frac{1}{\gamma_1-1}}+\left(\frac{\mu_2\xi_{T,d}\eta_{2,T-1}}{\beta^T}\right)^{\frac{1}{\gamma_2-1}} &= \delta_{T,d}. && (102)
\end{aligned}$$

Note that the system made of the four equations (89, 98, 101, 102) has four unknowns: $\nu_{1,T,u}$, $\nu_{1,T,d}$, $\xi_{T,u}$, $.\xi_{T,d}$[13] The solution can be obtained approximately as functions of $(\eta_{1,T-1}, \eta_{2,T-1}, \delta_{T-1})$. This gives us $X_{T,u}$ and $X_{T,d}$ as functions of $(\mu_1\eta_{1,T-1}, \mu_2\eta_{2,T-1}, \delta_{T-1})$.

Using the envelope theorem for consumptions and for the choice of ν, the derivatives of $X_{T,u}(\mu_1\eta_{1,T-1}, \mu_2\eta_{2,T-1}, \delta_{T-1})$ and $X_{T,d}(\mu_1\eta_{1,T-1}, \mu_2\eta_{2,T-1}, \delta_{T-1})$, including the change in the ξs, are

$$\begin{aligned}\partial_1 X_{T,u}(\mu_1\eta_{1,T-1}, \mu_2\eta_{2,T-1}, \delta_{T-1}) &= -\xi_{T,u} \times (1-\nu_{1,T,u})(c_{1,T,u} - \delta_{T,u}) \\ &\quad -\partial_1 \xi_{T,u}(\mu_1\eta_{1,T-1}, \mu_2\eta_{2,T-1}, \delta_{T-1}) \\ &\quad \times \left[\begin{array}{c} \mu_1\eta_{1,T-1}(1-\nu_{1,T,u})(c_{1,T,u} - \delta_{T,u}) \\ +\mu_2\eta_{2,T-1}c_{2,u} \end{array} \right]\end{aligned} \tag{103}$$

$$\begin{aligned}\partial_1 X_{T,d}(\mu_1\eta_{1,T-1}, \mu_2\eta_{2,T-1}, \delta_{T-1}) &= -\xi_{T,d} \times (1-\nu_{1,T-1,d})(c_{1,T,d} - \delta_{T,d}) \\ &\quad -\partial_1 \xi_{T,d}(\mu_1\eta_{1,T-1}, \mu_2\eta_{2,T-1}, \delta_{T-1}) \\ &\quad \times \left[\begin{array}{c} \mu_1\eta_{1,T-1}(1-\nu_{1,T,d})(c_{1,T,d} - \delta_{T,d}) \\ +\mu_2\eta_{2,T-1}c_{2,T,d} \end{array} \right]\end{aligned} \tag{104}$$

$$\begin{aligned}\partial_2 X_{T,u}(\mu_1\eta_{1,T-1}, \mu_2\eta_{2,T-1}, \delta_{T-1}) &= -\xi_{T,u} \times c_{2,u} \\ &\quad -\partial_2 \xi_{T,u}(\mu_1\eta_{1,T-1}, \mu_2\eta_{2,T-1}, \delta_{T-1}) \\ &\quad \times \left[\begin{array}{c} \mu_1\eta_{1,T-1}(1-\nu_{1,T,u})(c_{1,T,u} - \delta_{T,u}) \\ +\mu_2\eta_{2,T-1}c_{2,u} \end{array} \right]\end{aligned} \tag{105}$$

$$\begin{aligned}\partial_2 X_{T,d}(\mu_1\eta_{1,T-1}, \mu_2\eta_{2,T-1}, \delta_{T-1}) &= -\xi_{T,d} \times c_{2,T,d} \\ &\quad -\partial_2 \xi_{T,d}(\mu_1\eta_{1,T-1}, \mu_2\eta_{2,T-1}, \delta_{T-1}) \\ &\quad \times \left[\begin{array}{c} \mu_1\eta_{1,T-1}(1-\nu_{1,T,d})(c_{1,T,d} - \delta_{T,d}) \\ +\mu_2\eta_{2,T-1}c_{2,T,d} \end{array} \right].\end{aligned} \tag{106}$$

At the next step, one gets the derivatives keeping the ξs constant by notice that

$$\begin{aligned}\partial_1 X_{T,u} + \partial_2 X_{T,u} &= 0 - [\partial_1 \xi_{T,u} + \partial_2 \xi_{T,u}] \\ &\quad \times [\mu_1\eta_{1,T-1}(1-\nu_{1,T,u})(c_{1,T,u} - \delta_{T,u}) + \mu_2\eta_{2,T-1}c_{2,u}]\end{aligned} \tag{107}$$

so that

$$\mu_1\eta_{1,T-1}(1-\nu_{1,T,u})(c_{1,T,u} - \delta_{T,u}) + \mu_2\eta_{2,T-1}c_{2,u} = -\frac{\partial_1 X_{T,u} + \partial_2 X_{T,u}}{\partial_1 \xi_{T,u} + \partial_2 \xi_{T,u}} \tag{108}$$

and similarly

$$\mu_1\eta_{1,T-1}(1-\nu_{1,T,d})(c_{1,T,d} - \delta_{T,d}) + \mu_2\eta_{2,T-1}c_{2,d} = -\frac{\partial_1 X_{T,d} + \partial_2 X_{T,d}}{\partial_1 \xi_{T,d} + \partial_2 \xi_{T,d}}. \tag{109}$$

[13] Or, in the case of the last period, two equations (101, 102) in two unknowns: $\nu_{1,T-1}$, ξ_T

Hence

$$\begin{aligned}
\partial_1 X_{T,u}|_{\xi_{T,u}} &= -\xi_{T,u}\left(1-\nu_{1,T,u}\right)\left(c_{1,T,u}-\delta_{T,u}\right) &(110)\\
&= \partial_1 X_{T,u} - \partial_1 \xi_{T,u}\frac{\partial_1 X_{T,u}+\partial_2 X_{T,u}}{\partial_1 \xi_{T,u}+\partial_2 \xi_{T,u}}\\
\partial_1 X_{T,d}|_{\xi_{T,d}} &= -\xi_{T,d}\left(1-\nu_{1,T-1,d}\right)\left(c_{1,T,d}-\delta_{T,d}\right) &(111)\\
&= \partial_1 X_{T,d} - \partial_1 \xi_{T,d}\frac{\partial_1 X_{T,d}+\partial_2 X_{T,d}}{\partial_1 \xi_{T,d}+\partial_2 \xi_{T,d}}.
\end{aligned}$$

At $T-2$, the new X is

$$\overline{X}_{T-2} = \frac{1}{2}X_{T-1,u} + \frac{1}{2}X_{T-1,d} \tag{112}$$

where

$$\begin{aligned}
X_{T-1,u} &= \beta^{T-1}u_1\left(c_{1,T-1,u}\right) + \beta^{T-1}u_2\left(c_{2,T-1,u}\right) &(113)\\
&\quad -\mu_1\xi_{T-1,u}\eta_{1,T-2}\left(1-\nu_{1,T-1,u}\right)\left(c_{1,T-1,u}-\delta_{T-1,u}\right)\\
&\quad -\mu_2\xi_{T-1,u}\eta_{2,T-2}c_{2,T-1,u}\\
&\quad +\frac{1}{2}X_{T,u,u}\left(\mu_1\eta_{1,T-2}\left(1-\nu_{1,T-1,u}\right),\mu_2\eta_{2,T-1},\delta_{T-1}\right)\\
&\quad +\frac{1}{2}X_{T,u,d}\left(\mu_1\eta_{1,T-2}\left(1-\nu_{1,T-1,u}\right),\mu_2\eta_{2,T-1},\delta_{T-1}\right)\\
X_{T-1,d} &= \beta^{T}u_1\left(c_{1,T,d}\right) + \beta^{T}u_2\left(c_{2,T,d}\right)\\
&\quad -\mu_1\xi_{T,d}\eta_{1,T-2}\left(1-\nu_{1,T-1,d}\right)\left(c_{1,T,d}-\delta_{T,d}\right)\\
&\quad -\mu_2\xi_{T,d}\eta_{2,T-1}c_{2,T,d}\\
&\quad +\frac{1}{2}X_{T,d,u}\left(\mu_1\eta_{1,T-2}\left(1-\nu_{1,T-1,d}\right),\mu_2\eta_{2,T-1},\delta_{T-1}\right)\\
&\quad +\frac{1}{2}X_{T,d,d}\left(\mu_1\eta_{1,T-2}\left(1-\nu_{1,T-1,d}\right),\mu_2\eta_{2,T-1},\delta_{T-1}\right).
\end{aligned}$$

The Lagrangean associated is

$$\frac{1}{2}L_{T-1,u} + \frac{1}{2}L_{T-1,d} \tag{114}$$

where

$$\begin{aligned}
L_{T-1,u} &= X_{T-1,u} + \alpha_{1,T-2}\mu_1\left[\xi_{T-1,u}\eta_{1,T-2} - \xi_{T-1,u}\eta_{1,T-2}\left(1-\nu_{1,T-1,u}\right)\right] &(115)\\
L_{T-1,d} &= X_{T-1,d} + \alpha_{1,T-2}\mu_1\left[\xi_{T-1,d}\eta_{1,T-2} - \xi_{T-1,d}\eta_{1,T-2}\left(1-\nu_{1,T-1,d}\right)\right] &(116)
\end{aligned}$$

Differentiate wrt $\nu_{1,T-1,u}$ again keeping *all* ξs constant

$$\begin{aligned}
&\left.\tfrac{\partial L_{T-1,u}}{\partial \nu_{1,T-1,u}}\right|_{\xi_{T-1,u},\xi_{T,u,u},\xi_{T,u,d}} = \mu_1 \xi_{T-1,u} \eta_{1,T-2} \left(c_{1,T-1,u} - \delta_{T-1,u}\right) \\
&\qquad + \alpha_{1,T-2} \mu_1 \xi_{T-1,u} \eta_{1,T-2} \\
&\quad - \frac{1}{2} \left.\frac{\partial X_{T,u,u}\left(\mu_1 \eta_{1,T-2}\left(1-\nu_{1,T-1,u}\right), \mu_2 \eta_{2,T-1}, \delta_{T-1}\right)}{\partial\left(\mu_1 \eta_{1,T-1}\right)}\right|_{\xi_{T,u,u},\xi_{T,u,d}} \mu_1 \eta_{1,T-2} \\
&\quad - \frac{1}{2} \left.\frac{\partial X_{T,u,d}\left(\mu_1 \eta_{1,T-2}\left(1-\nu_{1,T-1,u}\right), \mu_2 \eta_{2,T-1}, \delta_{T-1}\right)}{\partial\left(\mu_1 \eta_{1,T-1}\right)}\right|_{\xi_{T,u,u},\xi_{T,u,d}} \mu_1 \eta_{1,T-2} \\
&= \; 0.
\end{aligned} \tag{117}$$

Similarly

$$\begin{aligned}
&\left.\tfrac{\partial L_{T-1,d}}{\partial \nu_{1,T-1,d}}\right|_{\xi_{T-1,d},\xi_{T,d,u},\xi_{T,d,d}} = \mu_1 \xi_{T-1,d} \eta_{1,T-2} \left(c_{1,T-1,d} - \delta_{T-1,d}\right) \\
&\qquad + \alpha_{1,T-2} \mu_1 \xi_{T-1,d} \eta_{1,T-2} \\
&\quad - \frac{1}{2} \left.\frac{\partial X_{T,d,u}\left(\mu_1 \eta_{1,T-2}\left(1-\nu_{1,T-1,d}\right), \mu_2 \eta_{2,T-1}, \delta_{T-1}\right)}{\partial\left(\mu_1 \eta_{1,T-1}\right)}\right|_{\xi_{T,d,u},\xi_{T,d,d}} \mu_1 \eta_{1,T-2} \\
&\quad - \frac{1}{2} \left.\frac{\partial X_{T,d,d}\left(\mu_1 \eta_{1,T-2}\left(1-\nu_{1,T-1,d}\right), \mu_2 \eta_{2,T-1}, \delta_{T-1}\right)}{\partial\left(\mu_1 \eta_{1,T-1}\right)}\right|_{\xi_{T,d,u},\xi_{T,d,d}} \mu_1 \eta_{1,T-2} \\
&= \; 0.
\end{aligned} \tag{118}$$

The system comprising these two first-order conditions, the price restriction and the two aggregate-resource constraints (of Planner 2) at time $t = T-1$ can be solved. This gives us $X_{T-1,u}$ and $X_{T-1,d}$ as functions of $(\mu_1\eta_{1,T-2}, \mu_2\eta_{2,T-2}, \delta_{T-2})$.

Using the envelope theorem for consumptions and for the choice of ν, the derivatives of $X_{T-1,u}\left(\mu_1\eta_{1,T-2}, \mu_2\eta_{2,T-2}, \delta\right.$ and $X_{T-1,d}\left(\mu_1\eta_{1,T-2}, \mu_2\eta_{2,T-2}, \delta_{T-2}\right)$, *including* the changes in $\xi_{T-1,u}$,$\xi_{T-1,d}$,$\xi_{T,u,u}$, $\xi_{T,u,d}$,$\xi_{T,d,u}$, $\xi_{T,d,d}$ are

$$\partial_1 X_{T-1,u}\left(\mu_1\eta_{1,T-2}, \mu_2\eta_{2,T-2}, \delta_{T-2}\right) = -\xi_{T-1,u}\left(1-\nu_{1,T-1,u}\right)\left(c_{1,T-1,u} - \delta_{T-1,u}\right) \tag{119}$$

$$\begin{aligned}
&-\partial_1 \xi_{T-1,u} \times \left[\mu_1 \eta_{1,T-2}\left(1-\nu_{1,T-1,u}\right)\left(c_{1,T-1,u} - \delta_{T-1,u}\right) + \mu_2 \eta_{2,T-2} c_{2,T-1,u}\right] \\
&+\frac{1}{2}\left\{\begin{array}{c} \left.\partial_1 X_{T,u,u}\left(\mu_1\eta_{1,T-2}\left(1-\nu_{1,T-1,u}\right), \mu_2\eta_{2,T-1}, \delta_{T-1}\right)\right|_{\xi_{T,u,u},\xi_{T,u,d}} \\ +\partial_1 \xi_{T,u,u} \frac{\partial_1 X_{T,u,u} + \partial_2 X_{T,u,u}}{\partial_1 \xi_{T,u,u} + \partial_2 \xi_{T,u,u}} \end{array}\right\} \\
&\qquad \times \left(1-\nu_{1,T-1,u}\right)
\end{aligned} \tag{120}$$

$$\begin{aligned}
&+\frac{1}{2}\left\{\begin{array}{c} \left.\partial_1 X_{T,u,d}\left(\mu_1\eta_{1,T-2}\left(1-\nu_{1,T-1,u}\right), \mu_2\eta_{2,T-1}, \delta_{T-1}\right)\right|_{\xi_{T,u,u},\xi_{T,u,d}} \\ +\partial_1 \xi_{T,u,d} \frac{\partial_1 X_{T,u,d} + \partial_2 X_{T,u,d}}{\partial_1 \xi_{T,d,u} + \partial_2 \xi_{T,u,d}} \end{array}\right\} \\
&\qquad \times \left(1-\nu_{1,T-1,u}\right)
\end{aligned} \tag{121}$$

$$
\begin{aligned}
\partial_2 X_{T-1,u}\left(\mu_1\eta_{1,T-2},\mu_2\eta_{2,T-2},\delta_{T-2}\right) &= -\xi_{T-1,u}c_{2,T-1,u} \\
-\partial_2\xi_{T-1,u}\times\left[\mu_1\eta_{1,T-2}\left(1-\nu_{1,T-1,u}\right)\left(c_{1,T-1,u}-\delta_{T-1,u}\right)+\mu_2\eta_{2,T-2}c_{2,T-1,u}\right] \\
+\frac{1}{2}\left\{\begin{array}{c}\partial_2 X_{T,u,u}\left(\mu_1\eta_{1,T-2}\left(1-\nu_{1,T-1,u}\right),\mu_2\eta_{2,T-1},\delta_{T-1}\right)\big|_{\xi_{T,u,u},\xi_{T,u,d}} \\ +\partial_2\xi_{T,u,u}\frac{\partial_1 X_{T,u,u}+\partial_2 X_{T,u,u}}{\partial_1\xi_{T,u,u}+\partial_2\xi_{T,u,u}}\end{array}\right\} \\
+\frac{1}{2}\left\{\begin{array}{c}\partial_2 X_{T,u,d}\left(\mu_1\eta_{1,T-2}\left(1-\nu_{1,T-1,u}\right),\mu_2\eta_{2,T-1},\delta_{T-1}\right)\big|_{\xi_{T,u,u},\xi_{T,u,d}} \\ +\partial_2\xi_{T,u,d}\frac{\partial_1 X_{T,u,d}+\partial_2 X_{T,u,d}}{\partial_1\xi_{T,d,u}+\partial_2\xi_{T,u,d}}\end{array}\right\}
\end{aligned} \tag{122}
$$

Summing

$$
\begin{aligned}
\partial_1 X_{T-1,u}+\partial_2 X_{T-1,u} &= 0-\left[\partial_1\xi_{T-1,u}+\partial_2\xi_{T-1,u}\right] \\
\times\left[\mu_1\eta_{1,T-2}\left(1-\nu_{1,T-1,u}\right)\left(c_{1,T-1,u}-\delta_{T-1,u}\right)+\mu_2\eta_{2,T-2}c_{2,T-1,u}\right] \\
+\frac{1}{2}\partial_1\xi_{T,u,u}\frac{\partial_1 X_{T,u,u}+\partial_2 X_{T,u,u}}{\partial_1\xi_{T,u,u}+\partial_2\xi_{T,u,u}}\left(1-\nu_{1,T-1,u}\right) \\
+\frac{1}{2}\partial_1\xi_{T,u,d}\frac{\partial_1 X_{T,u,d}+\partial_2 X_{T,u,d}}{\partial_1\xi_{T,d,u}+\partial_2\xi_{T,u,d}}\left(1-\nu_{1,T-1,u}\right) \\
+\frac{1}{2}\partial_2\xi_{T,u,u}\frac{\partial_1 X_{T,u,u}+\partial_2 X_{T,u,u}}{\partial_1\xi_{T,u,u}+\partial_2\xi_{T,u,u}} \\
+\frac{1}{2}\partial_2\xi_{T,u,d}\frac{\partial_1 X_{T,u,d}+\partial_2 X_{T,u,d}}{\partial_1\xi_{T,d,u}+\partial_2\xi_{T,u,d}}
\end{aligned} \tag{123}
$$

$$
\begin{aligned}
\partial_1 X_{T-1,u}+\partial_2 X_{T-1,u} &= 0-\left[\partial_1\xi_{T-1,u}+\partial_2\xi_{T-1,u}\right] \\
\times\left[\mu_1\eta_{1,T-2}\left(1-\nu_{1,T-1,u}\right)\left(c_{1,T-1,u}-\delta_{T-1,u}\right)+\mu_2\eta_{2,T-2}c_{2,T-1,u}\right] \\
+\left(\frac{1}{2}\partial_1\xi_{T,u,u}\left(1-\nu_{1,T-1,u}\right)+\frac{1}{2}\partial_2\xi_{T,u,u}\right)\frac{\partial_1 X_{T,u,u}+\partial_2 X_{T,u,u}}{\partial_1\xi_{T,u,u}+\partial_2\xi_{T,u,u}} \\
+\left(\frac{1}{2}\partial_1\xi_{T,u,d}\left(1-\nu_{1,T-1,u}\right)+\frac{1}{2}\partial_2\xi_{T,u,d}\right)\frac{\partial_1 X_{T,u,d}+\partial_2 X_{T,u,d}}{\partial_1\xi_{T,d,u}+\partial_2\xi_{T,u,d}}
\end{aligned} \tag{124}
$$

so that

$$
\begin{aligned}
\mu_1\eta_{1,T-2}\left(1-\nu_{1,T-1,u}\right)\left(c_{1,T-1,u}-\delta_{T-1,u}\right)+\mu_2\eta_{2,T-2}c_{2,T-1,u} &= -\frac{1}{\partial_1\xi_{T-1,u}+\partial_2\xi_{T-1,u}} \\
\left[\begin{array}{c}\partial_1 X_{T-1,u}+\partial_2 X_{T-1,u}-\left(\frac{1}{2}\partial_1\xi_{T,u,u}\left(1-\nu_{1,T-1,u}\right)+\frac{1}{2}\partial_2\xi_{T,u,u}\right) \\ \times\frac{\partial_1 X_{T,u,u}+\partial_2 X_{T,u,u}}{\partial_1\xi_{T,u,u}+\partial_2\xi_{T,u,u}}\end{array}\right. \\
\left.-\left(\frac{1}{2}\partial_1\xi_{T,u,d}\left(1-\nu_{1,T-1,u}\right)+\frac{1}{2}\partial_2\xi_{T,u,d}\right)\frac{\partial_1 X_{T,u,d}+\partial_2 X_{T,u,d}}{\partial_1\xi_{T,d,u}+\partial_2\xi_{T,u,d}}\right].
\end{aligned} \tag{125}
$$

Hence

$$\begin{aligned}
\partial_1 X_{T-1,u}\left(\mu_1\eta_{1,T-2},\mu_2\eta_{2,T-2},\delta_{T-2}\right)\big|_{\xi_{T-1,u},\xi_{T,u,u},\xi_{T,u,d}} &= \partial_1 X_{T-1,u}\left(\mu_1\eta_{1,T-2},\mu_2\eta_{2,T-2},\delta_{T-2}\right) \\
&\quad -\frac{\partial_1\xi_{T-1,u}}{\partial_1\xi_{T-1,u}+\partial_2\xi_{T-1,u}} \\
&\quad \times\Bigg[\ \partial_1 X_{T-1,u}+\partial_2 X_{T-1,u}-\left(\tfrac{1}{2}\partial_1\xi_{T,u,u}\left(1-\nu_{1,T-1,u}\right)+\tfrac{1}{2}\partial_2\xi_{T,u,u}\right) \\
&\qquad \times\tfrac{\partial_1 X_{T,u,u}+\partial_2 X_{T,u,u}}{\partial_1\xi_{T,u,u}+\partial_2\xi_{T,u,u}} \\
&\qquad -\left(\tfrac{1}{2}\partial_1\xi_{T,u,d}\left(1-\nu_{1,T-1,u}\right)+\tfrac{1}{2}\partial_2\xi_{T,u,d}\right) \\
&\qquad \times\tfrac{\partial_1 X_{T,u,d}+\partial_2 X_{T,u,d}}{\partial_1\xi_{T,d,u}+\partial_2\xi_{T,u,d}}\ \Bigg] \\
&\quad -\frac{1}{2}\partial_1\xi_{T,u,u}\frac{\partial_1 X_{T,u,u}+\partial_2 X_{T,u,u}}{\partial_1\xi_{T,u,u}+\partial_2\xi_{T,u,u}}\left(1-\nu_{1,T-1,u}\right) \\
&\quad -\frac{1}{2}\partial_1\xi_{T,u,d}\frac{\partial_1 X_{T,u,d}+\partial_2 X_{T,u,d}}{\partial_1\xi_{T,d,u}+\partial_2\xi_{T,u,d}}\left(1-\nu_{1,T-1,u}\right).
\end{aligned} \tag{126}$$

In (126), one readily observes that the compensation cannot work two periods ahead.