Fear of Recessions, Heterogenous Beliefs, and Stock Price Under/Over-Reaction*

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Abstract

Our purpose is to show how large difference of beliefs induced by the fear of a recession is amenable to large and persistent price responses to contemporaneous shocks. We construct a pure exchange economy populated by two agents who estimate strictly different models regarding the fundamental. In particular, one agent believes that recessions could occur whereas the second agent has smoother beliefs. We show that this setting is prone to generate disagreement whose persistence varies over a bearish phase. When economic conditions start to deteriorate, disagreement among agents significantly increases in a persistent fashion. Yet, after a sustained streak of bad news, disagreement gradually loses of its persistence. Contrary to the common wisdom that heterogeneous beliefs lead to large bets, we show that this pattern in disagreement ultimately causes the trading volume to dry up. As a result, price sensitivity to current shocks, along with persistence, is shown to be dramatically increased when compared with a model absent of heterogeneous beliefs, but with perceived recession risk. Using Google search data, we find empirical support for the main implication of the model.

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1 Introduction

Rare disasters have been proposed by Rietz (1988) and Barro (2006) as a potential explanation for some well-known financial puzzles. Yet, as noted by Mehra (2003), the type of disasters needed to resolve these empirical inconsistencies has not been observed in the United States for the past hundred years. Still, the mere possibility that a severely bad event could potentially occur carries other implications for asset prices. In particular, during bad times investors should be more reactive in updating their beliefs about the current state of the world and prices should, thus, exhibit different degrees of responsiveness to news shocks. This paper offers an investigation of this matter. Specifically, we analyze the feedback relation that exists between prices and investors beliefs *conditioning* on different phases of a bad economic episode.

Recent articles provide evidence that agents anticipate the occurrence of bad economic periods: Lustig and Verdelhan (2010) show, using Google search data and newspaper articles, that agents learn about the economy entering a recession before the official announcement is made[1]. Hence, agents become concerned about the possibility of a potential recession a long time before it actually occurs. Gourio, Siemer, and Verdelhan (2010) interpret the last financial crisis as the result of a surge in the perceived probability of a disaster: *"There is ample evidence that investors around the world feared a Great Depression scenario in the Fall of 2008"*. This interpretation is reminiscent of the feedback between prices and investors' beliefs mentioned above. More generally, as these evidences suggest, *recessions are such that they tend to be neglected in good times and to become of an increasing concern as the economy enters a bearish phase*. The latter is our premise: On the one hand, observed disasters affect aggregate consumption so rarely that it is hardly conceivable that prices reflect such concerns on a permanent basis. On the other hand, recessions, such as financial crises are sufficiently extreme for agents to bear them in mind, at least when the market conditions deteriorate. As in Veronesi (1999, 2000, 2004), we capture this idea by assuming that the expected growth rate of dividends—the fundamental—is unobservable and a subset of agents (say Agent A) learns about it postulating that it obeys a two state Markov chain. More precisely, Agent A reckons that the fundamental is stable most of the time (state 1), yet that it may drop *dramatically* as the Markov chain moves to state 2. This drop in the expected dividend growth should be seen as a rare disaster whose magnitude ranges from -10% to -15%. The transition probabilities are such that, on the one hand, a sharp decrease happens seldom while, on the other hand, a recovery is very likely

[1] In their study, they refer to official announcements as those made by NBER and OECD.

and Agent A's beliefs do not remain clouded by the threat of a recession for too long. Also, because the fundamental is unobservable, time-varying recession risk naturally and endogenously arises as the result of Agent A constantly reassessing the likelihood of recessions through the filtering process. This contrasts with studies in which the time-varying probability of a disaster is exogenously specified, as in Wachter (2011), for instance[2]. This precisely achieves our premise: Agent A features increasing concerns with respect to a substantial drop in the fundamental as economic conditions worsen. This view is also consistent with evidence such as in Bloom (2009) that recessions are characterized by a rise in uncertainty[3]: "*... the Depression was an example of the so-called "Peso problem," in the sense that there was legitimate uncertainty about whether the economic system would survive... Uncertainty about whether the "regime" had changed adds to the fundamental uncertainty reflected in past and future volatility of macroeconomic data.*"

Importantly, since the kind of recessions we are interested in are rare and unobservable, it is hard to assess their size or their frequency. Accordingly, as in Chen, Joslin, and Tran (2010), the difficulty of inferring magnitudes and frequencies of recessions gives rise to potential for differences of opinions among investors. However, unlike Chen, Joslin, and Tran (2010) who assume that agents are not learning, we assume that investors still need to estimate the model they postulate, despite the low frequency of recessions. Consequently, the low frequency of recessions along with their unobservable nature are highly suited for sentiment risk and heterogeneous beliefs. As it turns out, this aspect shall mainly drive our results: We introduce an agent (say Agent B) whose views regarding the current state of the economy are moderate. In particular, he conjectures that the economy transits from good to bad times in a smooth, cyclical manner. As a result, because Agent B reckons that the economy exhibits cycles, he adjusts his beliefs in a very persistent fashion. This induces a rich pattern in disagreement among Agent A and Agent B: When the economy first starts to show signs of a decline, Agent A is very fast to reassess the probability of a recession while Agent B's beliefs roughly remain unchanged. As a sustained streak of bad news is needed in order to tilt Agent B's expectations deep into a bearish cycle, disagreement is typically persistent at the beginning of a bad period and remains as such for some time. Yet, as economic conditions further deteriorate and as it, thus, becomes clear for both agents that the situation will most likely not get any worse, beliefs unanimously hint towards recovery and the disagreement process gradually loses of its persistence. Hence, although the fear

[2]See also Gourio (2008, 2011) or Gourio, Siemer, and Verdelhan (2010).

[3]This is a comment of Robert C. Merton in Schwert (1990).

of recessions is prone to generate a persistent disagreement, it progressively loses bite after a sequence of bad news. This rich persistence pattern in disagreement is fundamental to generate our results and does not arise when agents estimate close models, as in[4] Dumas, Kurshev, and Uppal (2009), for instance. In particular, we show that when Agent A is a representative agent, prices immediately respond to a current news shocks in a *non persistent* fashion. The reason for this rigid reaction is twofold: *i)* Since Agent A is a representative agent, he is required to hold the entire supply of the stock and, thus, no risk sharing is allowed. *ii)* Because Agent A is a representative agent, there is no potential for difference of beliefs. We find that difference of beliefs is critical to generate sluggish price responses during bad times. Specifically, the following mechanism prevails. As a series of bad shocks hits the economy, both agents adjusts their beliefs: Agent A anticipates a recession while Agent B anticipates a bearish cycle. Since Agent A is willing to sell and Agent B is not naturally inclined to take the counterparty to the trade in anticipation of a bearish cycle, trading volume dries up. This introduces persistent pressures on prices conveyed through the disagreement's behavior previously highlighted. We show that this persistence is characterized by a first phase of under-reaction where the market is slow to react to a current shock. This first stage of persistence entails continuation. The second phase is associated with over-reaction in which the market overshoots the effect of the shock. This fact has been documented[5] in Bondt and Thaler (1985); Barberis, Shleifer, and Vishny (1998) or Hong and Stein (1999). Another interesting implication of our model resides in the trading volume collapse during bad times: While heterogeneous beliefs are usually associated with agents taking bets against each other, the opposite situation turns out to happen in our model. Instead of exacerbating the extent of these bets, a large disagreement during bad times causes trading volume to collapse. This is due to both agents strongly expecting an increase in dividend growth to take place after a sufficiently long streak of bad news. Hence, although agents strongly disagree about the level of the fundamental growth rate during bad times, they both expect economic conditions to improve in a very near future. To the best of our knowledge, this implication, which arises because of the drastic difference in model estimation, is new.

Figure 1 represents the key modeling motivation of our paper. It depicts a measure of the analyst forecasts dispersion in the US from 1982 to 2007, and it shows that the dispersion among analysts jumps up in bad economic periods. Hence, the model implied disagreement dynamics described above capture precisely the pattern

[4] See also Brennan and Xia (2001), Scheinkman and Xiong (2003) or Xiong and Yan (2010).

[5] See also Moskowitz, Ooi, and Pedersen (2010) or Daniel, Hirshleifer, and Subrahmanyam (1998) for further references on the topic.

observed empirically.

[Insert Figure 1 about here.]

We provide a summary empirical investigation of the model's main implication and report that it finds appeal in the data. More precisely, we test whether prices are more responsive to news shocks in bad times. To that purpose, we suggest an empirical proxy for news shocks in Agent A's Markov chain model by looking at *Google search data*. In particular, we build an index of changes in Google search volumes for terms related to financial crisis and the likes. We, then, regress this constructed times series on the S&P 500 returns and document much stronger and significant price responses during bad times. Also, in the spirit of Farhi, Gabaix, Fraiberger, Ranciere, and Verdelhan (2009), our model carries implications for risk reversals. These strategies are particularly relevant for our model as they relate to hedging of downside risk. We show by means of simulations that the level of risk reversals rises during bad times. Stock returns and interest rates are found to be negatively related to risk reversals. Moreover, we analyze the model using a key aspect of equilibria with learning: When agents need to estimate hidden state dynamics, they use all the observable information they have at hand. As a result, a particular emphasis is put on observable shocks which show up in every equilibrium dynamics. Therefore, because agents cluster their attention on shocks they can observe, equilibrium quantities are expected to be highly sensitive to observable shocks. In particular, and most importantly, it is of major interest to assess how a bad shock, whose interpretation greatly differs among agents, propagates through the economy. Specifically, this allows to extract the under- and over-reaction effect through prices. This purpose is achieved through *impulse response functions* which have been widely used in Macroeconomics. In the case of the continuous-time setting that we shall consider, these are implemented through stochastic calculus of variations, namely Malliavin calculus. The importance of Malliavin calculus in analyzing models with learning was first, to the best of our knowledge, recognized by Dumas, Kurshev, and Uppal (2009). However, in the latter contribution, impulse response functions were used to investigate predictability. We go one step further and make use of these in order to investigate how a bad shock is amplified in equilibrium when a class of agents over-reacts in anticipation of a recession.

We briefly comment on the link of the present work with the literature: We significantly depart from the literature on disaster risk[6] in that we do not investigate the type of disasters that physically hit aggregate consumption. For the empirical

[6]This literature is so large that we do not even intend to make an exhaustive list. Instead, we refer the reader to Rietz (1988), Gabaix (2008), Farhi and Gabaix (2008) or Barro (2009).

relevance of disaster risk, we refer the reader to Barro and Ursúa (2008). In our model and in the spirit of Barro, Nakamura, Steinsson, and Ursua (2010), we rather model recessions as perceived by investors: investors fear that recessions might occur, whether or not they actually occur. That our results do not rely on the actual occurrence of disasters only makes our argument stronger. Moreover, because Agent A learns about the probability of recessions, our model relates to the literature featuring a time-varying probability of disasters as in Wachter (2011); Gourio (2008, 2011) or Gourio, Siemer, and Verdelhan (2010). However, unlike these references and much as in Veronesi (2004), we endogenize the probability of a recession through learning. Another main difference with respect to this strand of the literature is that we do not attempt to match particular asset moments such as the equity premium, stock price volatility or stock returns autocorrelation. In particular, most of these references use production economies along with recursive utilities, levered consumption and time-varying disaster risk in order to simultaneously solve empirical inconsistencies such as the equity premium puzzle, the excess volatility puzzle and the likes. In the present paper, we consider a pure exchange economy with time-additive preferences and merely attempt to demonstrate the feedback effect between asset prices and investors' beliefs. Our paper also relates to the heterogeneous beliefs literature[7] and particularly to Scheinkman and Xiong (2003), Dumas, Kurshev, and Uppal (2009) and Chen, Joslin, and Tran (2010). We contribute to this strand of the literature by offering a model in which agents estimate strictly different models: As previously argued, this produces a rich pattern in disagreement among investors. This rich heterogeneity is key to our results. Also, as in Chen, Joslin, and Tran (2010), our focus is to gauge the relevance of heterogeneous beliefs when recession risk is involved. Yet, on the one hand, we model recessions as being unobservable. On the other hand, our objective is very different in that we do not concentrate on the ability of our model to reach empirical moments. Finally, the way we analyze the model is partly borrowed from the Macroeconomics literature such as in Bloom (2009) or Gourio (2010), for instance. Not only for the mere fact that we use impulse responses but also because we condition on the different phases of the real business cycle (which is purely exogenous in our model) as in Lustig and Verdelhan (2010). Moreover, as in the latter reference, we also use Google search data in order to validate our model.

The paper is organized as follows: Section 2 exposes the benchmark model in which a representative agent fears that market recessions could occur. Section 3 introduces a second agent who has smooth forecasts about future dividend growth.

[7]See Brennan and Xia (2001), Scheinkman and Xiong (2003), or Xiong and Yan (2010).

Section 4 describes the model implications in details. Section 5 concludes. For convenience, proofs and computational details are reported in the Appendix.

2 A Benchmark Model: Representative Agent and Crash Risk

There is a single consumption good that serves as the *numéraire*. There are two assets: a locally riskless bond S^0 in zero net supply and a single risky asset S in positive net supply of one unit. The risky stock represents a claim to the strictly positive dividend process δ with dynamics

$$\mathrm{d}\delta_t = \delta_t f_t \mathrm{d}t + \delta_t \sigma_\delta \mathrm{d}W_t \tag{1}$$

for some exogenously given drift and volatility processes $(f, \sigma_\delta) \in \mathbb{R} \times \mathbb{R}_+$, and some initial value δ_0. W is a Brownian motion on the filtered probability space[8] $\left(\Omega, \mathscr{G}, \mathbb{P}^A\right)$. Finally, the probability measure $\mathbb{P}^A$ represents some reference probability measure.

The economy is populated by a representative agent who is assumed to have time-additive preferences represented by

$$U(c) \equiv \mathbb{E}\left[\int_0^\infty e^{-\rho s} \frac{c_s^{1-\alpha}}{1-\alpha} \mathrm{d}s\right]$$

where $\alpha > 0$ is the coefficient of relative risk aversion and $\rho > 0$ the subjective discount factor. This choice of preferences may appear to be restrictive: In Section H, we discuss how recursive utility may be introduced[9] and mention below how it may change our results. For the moment, it is sufficient to mention that more sophisticated preferences would significantly complicate the analysis without fundamentally altering our results. We thus assume this simpler type of utility for the sake of tractability.

Importantly, in our model, the expected growth rate of dividends, f, is assumed to be unobservable. As a result, it needs to be filtered out using the relevant observable information. Incidentally, the latter information is summarized by the stream of dividends. Furthermore, we make the central assumption that the representative agent proceeds information while having in mind that dividend growth could experience some dramatic decrease, yet with a very low probability. More precisely, we recognize

[8] $\mathscr{G}_t$ is the complete set of information available at time t. In other words, the filtration $\mathscr{G}$ is generated by the pair (f, W).

[9] In particular, we show how recursive utility may be introduced within the more general two agents framework of Section 3. The derivations for the present Benchmark case are obtained as a particular case of the computations exposed in Section H.

that fear of recessions should feature the following appealing property: *Crashes should be such that they tend to be neglected during good times and become of an increasing concern during bad times.* A continuous time two states Markov chain achieves this purpose: f can be any of the two possible values $f^d < 0$ and $f^s > 0$ where d stands for *disaster* and s for *steady*. The states are, accordingly, $f^M \equiv \left\{f^s, f^d\right\}$ with generator matrix

$$\Lambda = \begin{pmatrix} -\lambda & \lambda \\ \psi & -\psi \end{pmatrix}.$$

We can then make a perceived recession arbitrarily salient by adjusting the parameters λ and ψ. In particular, f jumps from the *steady* state to the *disaster* state with intensity λ, and from the *disaster* state to the *steady* state with intensity ψ. This means, in turn, that λ should be considered as of very small magnitude so as to reflect that recessions are rare events while ψ should be thought of as rather large in order to capture that a recession should not persist for a long time. Doing so, a two states Markov chain conveniently and parsimoniously attains our premise: On the one hand, setting λ sufficiently low, recessions are infrequent enough to be of a minor concern during periods characterized by sustained streaks of good news. On the other hand, for ψ sufficiently high, the prospects of recovery are strong enough for the representative agent's beliefs not to remain clouded by the threat of a recession. One could think of alternative ways of modeling fear of recessions, but this appears to be among the simplest and most appealing manners to incorporate this element in our model.

Notice that we depart from the literature[10] related to disaster risk in that our model and, thus, our results do not rely on the actual occurrence of recessions. Specifically, recessions do not cause a downside jump in the dividend stream itself, for, otherwise, they would be observable. Put differently, were recessions to physically hit aggregate output, there would be no potential for heterogeneous beliefs, an aspect which shall constitute the core of our analysis. Rather, in our model, recessions are modeled as a severe shock on the unobservable expected growth rate of dividend, and thus are possibly subject to disagreement among agents. Admittedly, by nature, recessions are, *i)* hard to estimate due to their very low frequency, hence raising the potential for dispersion of beliefs across the population of investors. Also, recessions are *ii)* sufficiently extreme to attract agents' consideration and provoke fear of recessions whenever market conditions deteriorate, irrelevant of whether or not they eventually occur. The present model offers such a specification of recessions.

[10]See Rietz (1988), Barro (2006), Farhi and Gabaix (2008), Gabaix (2008), Wachter (2011) and Chen, Joslin, and Tran (2010).

That our results do not depend on the actual realization of recessions only makes our argument stronger. This view makes the present framework closer to Barro, Nakamura, Steinsson, and Ursua (2010).

With this model in mind and conditional on his information set $\mathscr{F}_t = \sigma\left(\frac{\mathrm{d}\delta_s}{\delta_s} : 0 \leq s \leq t\right)$ at time t, the representative investor believes that f_t is f^s with probability π_t:

$$\pi_t = \mathrm{Prob}\left(f_t = f^s | \mathscr{F}_t\right).$$

The following proposition provides the dynamics of the filter.

Proposition 1. *The dynamics of the filter* $\widehat{f}_t = \mathbb{E}\left[f_t | \mathscr{F}_t\right] = \pi_t f^s + (1-\pi_t) f^d$ *are given by*

$$d\widehat{f}_t = (\lambda+\psi)\left(f_\infty - \widehat{f}_t\right) dt + \frac{1}{\sigma_\delta}\left(\widehat{f}_t - f^d\right)\left(f^s - \widehat{f}_t\right) d\widehat{W}_t \tag{2}$$

where

$$f_\infty = \lim_{t\to\infty} \mathbb{E}\left[f_t\right] = f^d + \frac{\psi}{\lambda+\psi}\left(f^s - f^d\right)$$

is the unconditional mean of the unobserved Markov chain and $\widehat{W}$ *the so-called innovation process*[11] *which is defined by*

$$\widehat{W}_t \equiv \frac{1}{\sigma_\delta}\int_0^t \left(\frac{d\delta_s}{\delta_s} - \widehat{f}_s\right) ds.$$

Proof. See Lipster and Shiryaev (2001). Q.E.D.

The filter described in Proposition 1 reflects the representative agent's beliefs that the economy is stable most of the time. Nonetheless, it incorporates the feature discussed above that some extremely severe events could occur with some very low probability. This suggests that the filter, $\widehat{f}$, moves away from the neighborhood of f^s only if there is an important enough succession of bad shocks in the dividend stream, δ. Consistent with the previous discussion, this is precisely how we want the filtered Markov chain to behave.

2.1 Equilibrium in the Benchmark Model

Since markets are complete, we can solve for the stock price in the spirit of Cox and Huang (1989). Using the first order condition of the usual static problem, we can show that the state price density, ξ, satisfies

$$\xi_t \equiv e^{-\int_0^t r_s^f \mathrm{d}s - \frac{1}{2}\int_0^t \theta_s^2 \mathrm{d}s - \int_0^t \theta_s \mathrm{d}\widehat{W}_s} = e^{-\rho t}\left(\frac{\delta_t}{\delta_0}\right)^{-\alpha}$$

[11]In other words, the innovation process $\widehat{W}$ is a $(\mathbb{P}^A, \mathscr{F})$-Brownian motion.

where r^f is the risk free rate and θ the market price of risk. Assuming that the horizon is infinite, the stock price is defined by

$$S_t|_{M.C.} \equiv \mathbb{E}\left[\int_t^\infty \frac{\xi_s}{\xi_t}\delta_s \mathrm{d}s \middle| \mathscr{F}_t\right]$$

and is characterized in the following proposition.

Proposition 2. *Define the matrix A as follows*

$$A = -\Omega - (1-\alpha)diag(f^M) + (\rho + \frac{1}{2}\alpha(1-\alpha)\sigma_\delta^2)\mathbf{Id}_2$$

where $\mathbf{Id}_2$ and $\mathbf{1}_2$ represent a two by two identity matrix and a 2-dimensional vector of ones, respectively. Then, when the expected growth rate of the dividend process is assumed to follow a two state Markov chain, the corresponding price dividend-ratio satisfies

$$\left.\frac{S_t}{\delta_t}\right|_{M.C.} = \pi_t H_1 + (1-\pi_t)H_2 = \frac{\widehat{f}_t - f^d}{f^s - f^d}H_1 + \left(1 - \frac{\widehat{f}_t - f^d}{f^s - f^d}\right)H_2$$

where $H \equiv \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = A^{-1}\mathbf{1}_2.$

Proof. The proof of the above statement is provided in Veronesi (2000). One may refer to Appendix A for the derivation of the price of a single-dividend claim. Q.E.D.

Since the stock price dynamics are, by definition, given by

$$\mathrm{d}S_t|_{M.C.} = S_t|_{M.C.}\left(\mu_t \mathrm{d}t + \sigma_t \mathrm{d}\widehat{W}_t\right) - \delta_t \mathrm{d}t,$$

the stock return volatility, σ, satisfies

$$\sigma_t = \frac{\partial S_t}{\partial x_t}\frac{\mathrm{diff}(x_t)^\top}{S_t}$$

where $x \equiv (\delta, \widehat{f})$ represents the state vector, diff(.) the diffusion operator and $(.)^\top$ the transpose operator. On the other hand the risk premium, RP, simply satisfies

$$RP_t = \theta_t \sigma_t.$$

For the purpose of the analysis of the next subsection, we compute the impulse response of the single-dividend paying stock price, $S_u^T \equiv \mathbb{E}\left[\frac{\xi_T}{\xi_u}\delta_T \middle| \mathscr{F}_u\right]$ with respect to the Brownian motion, $\widehat{W}$. This allows to better understand how the observable shock $\widehat{W}$ affects the price. This computation is done by means of Malliavin calculus[12]. The same tool also permits to decompose the volatility[13], σ. The following proposition characterizes the decomposition of the stock return volatility, σ_t.

Proposition 3. *The stock return volatility, σ_t, is decomposed as*

$$\begin{aligned} \sigma_t &= \theta_t + \frac{\phi_t}{\xi_t S_t} \\ &= \sigma_\delta + \frac{1-\alpha}{\xi_t S_t} \mathbb{E}\left[\int_t^\infty \xi_s \delta_s \int_t^s \mathscr{D}_t f_u \, du \, ds \middle| \mathscr{F}_t \right] \end{aligned}$$

because

$$\phi_t = \mathbb{E}\left[\int_t^\infty \xi_s \delta_s \left(\frac{\mathscr{D}_t \delta_t}{\delta_t} + \frac{\mathscr{D}_t \xi_t}{\xi_t} + (1-\alpha) \int_t^s \mathscr{D}_t f_u \, du \right) ds \middle| \mathscr{F}_t \right].$$

Notice that the variables $\frac{\mathscr{D}_t \delta_t}{\delta_t}$ and $-\frac{\mathscr{D}_t \xi_t}{\xi_t}$ are equal to σ_δ and $\theta_t = \alpha \sigma_\delta$, respectively.

Proof. Since the above statement is a particular case of Proposition 9, we do not provide any additional details on it. Q.E.D.

The following proposition characterizes the time-t expectation of the Malliavin derivative $\mathscr{D}_t S_u^T$ for $t \le u \le T$. Notice that S_u^T represents the single-dividend paying stock price at time u for a dividend to be paid at time T.

Proposition 4. *The time-t expectation of the Malliavin derivative $\mathscr{D}_t S_u^T$ satisfies*

$$\begin{aligned} \mathbb{E}\left[\mathscr{D}_t S_u^T \middle| \mathscr{F}_t \right] &= \frac{\mathscr{D}_t \delta_t}{\delta_t} \mathbb{E}\left[S_u^T \middle| \mathscr{F}_t \right] \\ &\quad + \mathbb{E}\left[\frac{\xi_T}{\xi_u} \delta_T \left((1-\alpha) \int_t^T \mathscr{D}_t \widehat{f}_s \, ds + \alpha \int_t^u \mathscr{D}_t \widehat{f}_s \, ds \right) \middle| \mathscr{F}_t \right]. \end{aligned}$$

Proof. Since the above statement is a particular case of Proposition 10, we do not provide any additional details on the computation of the Malliavin derivatives. See Appendix A for the computation of the price S_t^T of the single-dividend claim. Q.E.D.

Both, the volatility and the impulse response of the stock price in Propositions 3 and 4 contain two terms: The first term represents the direct short-term effect of a

[12] Definitions and details can be found in Malliavin and Thalmaier (2005).

[13] More precisely, the martingale representation theorem along with the Clark-Ocone theorem allow to obtain a clean decomposition of the volatility.

shock today through the direct channel of dividends. More interestingly, the second term is associated with anticipated future moves in the filtered dividend growth. In other words, it pertains to the long-term impact on prices of a shock today which is conveyed through learning. Given the representative agent's potentially extreme views on possible future outcomes, it is of particular interest to investigate how a contemporaneous shock is predicted to change the representative agent's expectations regarding future growth. We postpone an extensive discussion of this issue to Section 4 where we analyze the general model.

2.2 Results

Figure 2 plots the price responses with respect to a shock in the observable Brownian $\widehat{W}$ along with a proxy for the stock price diffusion[14]. The calibration used for this exercise may be found in Table 2. In Subsection 4.1.1, we provide an illustration of what this calibration implies for conditional moments of interest. In order to gather some more concrete intuition, in Subsection 4.1.2 we provide an extensive discussion of the empirical counterpart to a Brownian shock in the context of our model.

[Insert Figure 2 about Here.]

Except for the magnitude, graphs pertaining to price impulse responses, i.e. Figures 2a, 2b, 2c, and 2d exhibit the same pattern. The latter figures illustrate the expected impact at time $u \geq t$ of a shock in the observed Brownian $\widehat{W}$ at date t. Two important elements need to be emphasized with respect to these patterns: *i)* On the one hand, it may be observed that the price experiences an immediate negative jump and then quickly and monotonically reverts back to some equilibrium path. In other words, it exhibits almost no persistence. There are two reasons for this: First, the representative agent being required to hold the entire supply of the stock, the stock price reacts in a rigid fashion because no risk sharing is made possible. Second, in the absence of other agents and, thus, potential difference of beliefs, it is delicate to generate persistence in price responses. We shall turn to this matter in great details in the next section. *ii)* On the other hand, the effect of a shock in the Brownian today is maximized when the uncertainty is the greatest. That is, when $\widehat{f} = -0.03$ or, equivalently, when $\pi = \frac{1}{2}$. This may be seen from Figures 2b and 2e. At that point, the representative agent attributes equal weights to the steady and

[14]We observe that, for relatively large maturities, the volatility of a single-dividend paying stock is close to the volatility of the stock. Accordingly, it is almost equivalent to perform numerical experiments on the single-dividend paying stock keeping its maturity T distant to the present date t. For simulation purposes, we used $T = 50$.

the disaster state. As a result, a move in the Brownian provides a precious hint as to the direction of the future growth rate. Accordingly, the volatility and the impact of a shock are maximized. This fact has already been documented in Veronesi (2000) and David (2008).

Moreover, because the filtered growth rate $\widehat{f}$ evolves within a closed domain, when it reaches its upper and lower bounds, i.e. f^s and f^d, a shock in the Brownian does not significantly affect $\widehat{f}$ except through the direct channel of the dividend diffusion σ_δ. This fact is clearly illustrated by the magnitude of the graph in Figure 2d. Additionally, notice that Figures 2a and 2c are very similar: They both represent the symmetric cases where the Markov chain is 2% lower than f^s and 2% higher than f^d, respectively.

Finally, the direction of the effect may appear to be slightly puzzling: In particular, one would intuitively expect a positive shock today to produce an immediate increase in the price. When constant relative risk aversion preferences are involved, however, this relation turns out to be actually reversed. More precisely, the following mechanism prevails: When $\widehat{f}$ reaches its bottom, the representative agent knows that the dividend rate, on the one hand, cannot experience any further decrease, and, on the other hand, will move upwards with an extremely high probability. Given the time-additive feature of utility, the agent anticipates that a negative shock today only harms his current consumption and does not affect his future consumption plan. Hence, at a bottom state, the stock price should be the highest in order to reflect the dividend rate's imminent increase. The converse is true when $\widehat{f}$ is large. That is, because of anticipation of the future behavior of the expected dividend growth process along with time-additive constant relative risk aversion preferences, the price attains its maximum and minimum at the lower and upper bound, respectively. This rationale may be turned over if we allow preferences to show some sufficient degree of recursion. In particular, if we assume Epstein-Zin utility with an elasticity of intertemporal substitution larger than 1, the agent is sufficiently worried that a negative shock today might impact his future consumption plan for the price to drop. We refer the reader to Gourio (2010) or Wachter (2011) for a careful analysis of this matter. Yet, we choose to hold on to time-additive preferences for the rest of the analysis. The reasons are threefold: First, even using recursive preferences, we would have to resort to an elasticity of intertemporal substitution close to 2 in order for the desired effect to take place. This is, however, typically not supported by the data. Second, recursive preferences would significantly complicate the analysis. Still, we expose how this feature may be introduced in the model in Section H. Third, since this only affects the sign of the effect, our results and the point we want to make are preserved either way. As far as the Benchmark model is concerned, our goal is merely to show that prices do not respond persistently to an observable shock today.

3 Introducing Moderate Beliefs

We now consider the model of Section 2 with the representative agent (Agent A) and introduce a second agent (Agent B) who shares more moderate views regarding future outcomes. In particular, Agent B thinks that the expected growth rate of dividend exhibits cycles. Agent B, in contrast to Agent A, conjectures that the economy moves relatively smoothly from good to bad periods. Importantly, Agent B reckons that sharp declines in output growth do not occur. Specifically, he assumes that the unobservable growth rate of dividend follows an Ornstein-Uehlenbeck process. As a consequence, both agents disagree about the hidden dynamics of the dividend's drift, f. For convenience, we shall write the equilibrium under the probability measure that reflects Agent A's views, $\mathbb{P}^A$.

Given the setup described above, we anticipate the following mechanism: when a succession of bad shocks occurs, investor A adjusts his beliefs and reassesses the probability that the economy switches to the disaster state upwards. Anticipating the occurrence of a recession, Agent A is willing to massively leverage himself in the stock and lends the proceeds of the short sale. On the other hand, Agent B who also realizes that the economy enters a bad period, anticipates a bearish cycle and, as an optimal reaction, is eager to short the stock too. Consequently, even though the disagreement between the two agents increases, trading volume dries up. This is an interesting implication of our model: While heterogeneous beliefs are usually associated with agents taking bets against each other, we shall demonstrate that the opposite situation turns out to happen in the present model. Instead of exacerbating the extent of these bets, a large disagreement during bad times causes trading volume to collapse. This is due to both agents strongly expecting an increase in dividend growth to take place very soon. Consistently, a series of bad shocks should produce price pressures that should be stronger than in the representative agent case of Section 2. Indeed, since the disagreement is tremendous in some states of the economy, the impact on prices should be larger and more persistent than in the representative agent setup.

Agent B has the following model in mind.

$$\begin{aligned} \mathrm{d}\delta_t &= f_t^B \delta_t dt + \sigma_\delta \delta_t \mathrm{d}W_t^B \\ \mathrm{d}f_t^B &= \kappa \left(\overline{f} - f_t^B \right) \mathrm{d}t + \sigma_f \mathrm{d}W_t^f \end{aligned} \tag{3}$$

where κ and $\overline{f}$ represent the mean-reversion speed and the long-term mean, respectively. Moreover, W^f and W^B are two independent $(\mathbb{P}^B, \mathscr{G})-$Brownian motions. The following proposition provides the dynamics of the filter under Agent B's probability measure.

Proposition 5. *The filter $\widehat{f}_t^B = \mathbb{E}^{\mathbb{P}^B}\left[f_t^B | \mathscr{F}_t\right]$ satisfies*

$$d\widehat{f}_t^B = \kappa\left(\overline{f} - \widehat{f}_t^B\right) dt + \frac{\gamma}{\sigma_\delta} d\widehat{W}_t^B \tag{4}$$

where $\gamma = \sqrt{\sigma_\delta^2\left(\sigma_\delta^2\kappa^2 + \sigma_f^2\right)} - \kappa\sigma_\delta^2$ denotes the steady state variance of the filter and $\widehat{W}^B$ is a $(\mathbb{P}^B, \mathscr{F})-$Brownian motion.

Proof. See Appendix B. Q.E.D.

The dynamics of the filter in Proposition 5 reflects Agent B's smooth views that dividend growth mean-reverts around its long-term mean $\bar{f}$. These views are moderate in the sense that f never wanders significantly away from its long-term mean. Also, because of f's cyclical dynamics, Agent B expects f to go down whenever $f > \bar{f}$ or to go up whenever $f < \bar{f}$.

Define Agent B's probability measure as $\mathbb{P}^B(C) := \mathbb{E}\left[\eta_t \mathbf{1}_C\right]$, $C \in \mathscr{F}_t$ for all $0 \leq t < \infty$. Let the Radon-Nikodym derivative η satisfy

$$\left.\frac{\mathrm{d}\mathbb{P}^B}{\mathrm{d}\mathbb{P}^A}\right|_{\mathscr{F}_t} \equiv \eta_t = \exp\left[-\frac{1}{2}\int_0^t \frac{g_u^2}{\sigma_\delta^2}\mathrm{d}u - \int_0^t \frac{g_u}{\sigma_\delta}\mathrm{d}\widehat{W}_u\right]$$

where $g_t = \widehat{f}_t - \widehat{f}_t^B$ represents the disagreement between the two agents. Appealing to Girsanov's Theorem[15], the following relationship between the two Brownian motions prevails

$$\widehat{W}_t^B = \widehat{W}_t + \int_0^t \frac{g_u}{\sigma_\delta}\mathrm{d}u.$$

An application of Itô's lemma shows that g satisfies

$$\begin{aligned}\mathrm{d}g_t = &\left(\left[-\kappa\overline{f} + (\lambda + \psi) f_\infty\right] - \left(\kappa + \frac{\gamma}{\sigma_\delta^2}\right) g_t + (\kappa - (\lambda + \psi))\,\widehat{f}_t\right)\mathrm{d}t \\ &+ \frac{1}{\sigma_\delta}\left(\left(\widehat{f}_t - f^d\right)\left(f^s - \widehat{f}_t\right) - \gamma\right)\mathrm{d}\widehat{W}_t.\end{aligned} \tag{5}$$

A few words regarding the dynamics of the disagreement in (5) are in order. First, these dynamics are richer than the dynamics that would obtain in a model with agents estimating close models. For instance, in Dumas, Kurshev, and Uppal (2009),

[15] Assuming that g satisfies the Novikov condition.

the disagreement process mean-reverts around 0, thus reflecting that agents agree in a persistent fashion. In the case of the present model, agents disagree in a less trivial manner: When $\widehat{f}$ is close to its unconditional mean f_∞ and $\widehat{f}^B$ close to its long-term mean $\bar{f}$, the persistence induced by Agent B's model dominates. As a result, the drift of the disagreement process exhibits significant persistence. However, and this will play a central role in the analysis, persistence in the disagreement washes out as soon as beliefs head closer to f^s and f^d. This diversity in the persistence pattern mainly drives our results.

3.1 Optimal Consumption Plans

Because market completeness is preserved, we can solve both agents' consumption-portfolio problem using the martingale approach: On the one hand, Agent A solves

$$\max_c \mathbb{E}\left[\int_0^\infty e^{-\rho t}\frac{c_t^{1-\alpha}}{1-\alpha}\mathrm{d}t\right] + \phi\left(X_0 - \mathbb{E}\left[\int_0^\infty \xi_t c_t \mathrm{d}t\right]\right)$$

where ϕ denotes the Lagrangian multiplier of Agent A's static budget constraint. On the other hand, Agent B solves an analogous problem except for incorporating a change of measure towards Agent A's expectations:

$$\max_{c_B} \mathbb{E}\left[\int_0^\infty \eta_t e^{-\rho t}\frac{c_{Bt}^{1-\alpha}}{1-\alpha}\mathrm{d}t\right] + \phi_B\left(X_{B,0} - \mathbb{E}\left[\int_0^\infty \xi_t c_{Bt} \mathrm{d}t\right]\right).$$

The first order conditions of both problems respectively are

$$c_t = \left(\phi e^{\rho t}\xi_t\right)^{-\frac{1}{\alpha}}$$
$$c_{Bt} = \left(\frac{\phi_B e^{\rho t}\xi_t}{\eta_t}\right)^{-\frac{1}{\alpha}}.$$

Clearing the market yields an expression for the state-price density, ξ,

$$\xi_t = e^{-\rho t}\delta_t^{-\alpha}\left[\left(\frac{\eta_t}{\phi_B}\right)^{\frac{1}{\alpha}} + \left(\frac{1}{\phi}\right)^{\frac{1}{\alpha}}\right]^\alpha.$$

We can then use this expression to rewrite consumptions as

$$c_t = \omega\left(\eta_t\right)\delta_t$$
$$c_{Bt} = \left(1-\omega\left(\eta_t\right)\right)\delta_t$$

where $\omega(\eta_t)$ is the share of consumption of Agent A. It satisfies

$$\omega(\eta_t) = \frac{\left(\frac{1}{\phi}\right)^{\frac{1}{\alpha}}}{\left(\frac{\eta_t}{\phi_B}\right)^{\frac{1}{\alpha}} + \left(\frac{1}{\phi}\right)^{\frac{1}{\alpha}}}.$$

Finally, following Dumas, Kurshev, and Uppal (2009), we assume that the coefficient of relative risk aversion α is an integer. As a result, the single-dividend paying stock price, S_t^u, is written

$$\begin{aligned} S_t^u &= \frac{1}{\xi_t}\mathbb{E}\left[\xi_u \delta_u \,|\mathscr{F}_t\right] \qquad (6) \\ &= e^{-\rho(u-t)}\omega(\eta_t)^{\alpha}\,\delta_t^{\alpha}\sum_{j=0}^{\alpha}\begin{pmatrix}\alpha\\ j\end{pmatrix}\left(\frac{1}{\eta_t}\right)^{\frac{j}{\alpha}}\left(\frac{1-\omega(\eta_t)}{\omega(\eta_t)}\right)^{j}\mathbb{E}\left[\eta_u^{\frac{j}{\alpha}}\delta_u^{1-\alpha}\middle|\mathscr{F}_t\right]. \end{aligned}$$

Further details regarding Equation (6) can be found in Dumas, Kurshev, and Uppal (2009).

3.2 Equilibrium Stock Price

Using Equation (6), standard computations show that the stock price-dividend ratio satisfies

$$\frac{S_t}{\delta_t} = \omega(\eta_t)^{\alpha}\sum_{j=0}^{\alpha}\begin{pmatrix}\alpha\\ j\end{pmatrix}\left(\frac{1-\omega(\eta_t)}{\omega(\eta_t)}\right)^{j}\mathbb{E}\left[\int_t^{\infty} e^{-\rho(u-t)}\left(\frac{\eta_u}{\eta_t}\right)^{\frac{j}{\alpha}}\left(\frac{\delta_u}{\delta_t}\right)^{1-\alpha}du\middle|\mathscr{F}_t\right]. \qquad (7)$$

The expectation pertaining to Equation (7) is characterized in the following proposition.

Proposition 6. *The function F_t^j of $\widehat{f}_t$ and g_t defined by*

$$F_t^j\left(\widehat{f}_t, g_t\right) = \mathbb{E}\left[\int_t^{\infty} e^{-\rho(u-t)}\left(\frac{\eta_u}{\eta_t}\right)^{\frac{j}{\alpha}}\left(\frac{\delta_u}{\delta_t}\right)^{1-\alpha}du\middle|\mathscr{F}_t\right]$$

satisfies the partial differential equation[16]

$$\widetilde{\mathscr{L}}^{\widehat{f},g}F^j + X^jF^j + 1 = 0 \qquad (8)$$

where $\widetilde{\mathscr{L}}$ denotes the infinitesimal generator of $\left(\widehat{f}, g\right)$ under $\widetilde{\mathbb{P}}$.

[16] As proved in David (2008), the boundary conditions are absorbing in both the $\widehat{f}-$ and the $g-$dimensions.

Proof. See Appendix C. Q.E.D.

Notice that (8) admits a closed-form solution in the particular case for which $j = 0$ and an up to an integral closed-form solution for $j = \alpha$. These corresponds to the Lucas economies in which the representative agent has a two states Markov chain and an Ornstein-Uehlenbeck model in mind, respectively. Indeed, by imposing j to be zero the expectation in (7) becomes

$$\mathbb{E}\left[\int_t^\infty e^{-\rho(u-t)}\left(\frac{\delta_u}{\delta_t}\right)^{1-\alpha} \mathrm{d}u \middle| \mathscr{F}_t\right].$$

When $j = \alpha$ this term is

$$\mathbb{E}\left[\int_t^\infty e^{-\rho(u-t)}\left(\frac{\eta_u}{\eta_t}\right)\left(\frac{\delta_u}{\delta_t}\right)^{1-\alpha} \mathrm{d}u \middle| \mathscr{F}_t\right] = \mathbb{E}^{\mathbb{P}^B}\left[\int_t^\infty e^{-\rho(u-t)}\left(\frac{\delta_u}{\delta_t}\right)^{1-\alpha} \mathrm{d}u \middle| \mathscr{F}_t\right].$$

Since the case for which $\alpha = 0$ was already computed in Proposition 2, we are left with the computation of the price-dividend ratio under the assumption that the agent has a mean-reverting process in mind. The following proposition characterizes the required quantity.

Proposition 7. *When the expected growth rate of the dividend process is assumed to follow an Ornstein-Uehlenbeck process, the corresponding price dividend-ratio is*

$$\left.\frac{S_t}{\delta_t}\right|_{O.U.} = \int_0^\infty e^{-\rho\tau+\alpha(\tau)+\beta_2(\tau)\widehat{f}_t^B} d\tau = \int_0^\infty e^{-\rho\tau+\alpha(\tau)+\beta_2(\tau)\left(\widehat{f}_t-g_t\right)} d\tau$$

where the functions $\alpha(\tau)$ and $\beta_2(\tau)$ are the solutions to a set of Ricatti equations.

Proof. See Appendix D for the solutions of $\alpha(\tau)$ and $\beta_2(\tau)$. Q.E.D.

In order to obtain all the terms contained in the expression for the price-dividend ratio, we still need to compute the interior terms, namely those corresponding to $j \neq 0$ and $j \neq \alpha$. To do so, we numerically solve the partial differential equation in (8) by using the Chebyshev collocation method described in Judd (1998). The following proposition provides an approximate expression for the price-dividend ratio associated with the model with risk sharing.

Proposition 8. *The approximated price-dividend ratio satisfies*

$$\frac{S_t}{\delta_t} \approx \omega(\eta_t)^\alpha \left.\frac{S_t}{\delta_t}\right|_{M.C.} + \omega(\eta_t)^\alpha \sum_{j=1}^{\alpha-1} \begin{pmatrix} \alpha \\ j \end{pmatrix} \left(\frac{1-\omega(\eta_t)}{\omega(\eta_t)}\right)^j P^j + (1-\omega(\eta_t))^\alpha \left.\frac{S_t}{\delta_t}\right|_{O.U.}$$

where P^j stands for Chebyshev polynomial with index $j = 1, ..., \alpha - 1$. All the details concerning these polynomials are given in the proof.

Proof. See Appendix E. Q.E.D.

Finally, using the definition of the state-price density, the risk free rate, r^f, and the market price of risk, θ, are

$$r_t^f = \rho + \alpha\widehat{f}_t - \alpha\left(1-\omega(\eta_t)\right)g_t + \frac{1}{2}\left(\frac{\alpha-1}{\alpha\sigma_\delta^2}\omega(\eta_t)\left(1-\omega(\eta_t)\right)g_t^2 - \alpha(\alpha+1)\sigma_\delta^2\right)$$
$$\theta_t = \alpha\sigma_\delta + \frac{\left(1-\omega(\eta_t)\right)}{\sigma_\delta}g_t.$$

The stock return volatility is now given by

$$\sigma_t = \frac{\partial S_t}{\partial x_t}\frac{\text{diff}(\delta_t, \widehat{f}_t, g_t)^\top}{S_t}$$

where $x \equiv (\delta, \widehat{f}, g, \eta)$ represents the state vector, diff(.) and $(.)^\top$ are defined as in Section 2. Since we are still dealing with a one-dimensional Brownian Motion $\widetilde{W}$ only, the market price of risk and the stock return volatility are scalars. As a result, the risk premium, RP, still satisfies $RP_t = \theta_t\sigma_t$.

3.3 Optimal Portfolio Choice and Trading Volume

In order to characterize the way both agents invest in the stock market, we first need to compute Agent A's wealth, V, which is defined by

$$V_t = \mathbb{E}\left[\int_t^\infty \frac{\xi_u}{\xi_t}c_u \text{d}u \middle| \mathscr{F}_t\right].$$

Following the same steps as in Dumas, Kurshev, and Uppal (2009), the wealth of Agent A can be rewritten as

$$V_t = \delta_t\omega\left(\eta_t\right)^\alpha \sum_{j=0}^{\alpha-1}\begin{pmatrix}\alpha-1\\ j\end{pmatrix}\left(\frac{1-\omega\left(\eta_t\right)}{\omega\left(\eta_t\right)}\right)^j$$
$$\times\, \mathbb{E}\left[\int_t^\infty e^{-\rho(u-t)}\left(\frac{\eta_u}{\eta_t}\right)^{\frac{j}{\alpha}}\left(\frac{\delta_u}{\delta_t}\right)^{1-\alpha}\text{d}u \middle| \mathscr{F}_t\right].$$

Notice that the only difference between the wealth and the price is that the binomial sum upper bound is $\alpha - 1$ for the wealth as compared to α for the price. Consequently, it can be computed in exactly the same fashion as the price. Furthermore, the wealth dynamics are defined by

$$dV_t = (\ldots)\mathrm{d}t + \sigma_t Q_t \mathrm{d}\widehat{W}_t$$

where Q represents the amount invested by Investor A in the risky asset. That is, it satisfies

$$Q_t = \frac{\partial V_t}{\partial x_t} \frac{\mathrm{diff}(\delta_t, \widehat{f}_t, g_t, \eta_t)^\top}{\sigma_t}.$$

Finally, we are left with the computation of the trading volume which shall play an important role in the analysis. Such a measure is hard to pin down within a continuous-time setting. We achieve this purpose by following Xiong and Yan (2010): The trading volume is measured by the diffusion of the number of stocks Agent A holds. In other words, the trading volume is defined as

$$\mathrm{Volume}_t = \frac{\partial q_t}{\partial x_t} \mathrm{diff}(\delta_t, \widehat{f}_t, g_t, \eta_t)^\top \tag{9}$$

where $q_t = \frac{Q_t}{S_t}$ is the number of stocks held by Agent A at time t.

4 Model Implications

The model exposed in Section 3 carries two main implications: On the one hand, in the Benchmark model of Section 2, we argued that stock price responses to contemporaneous shocks exhibit almost no persistence. The absence of risk sharing and difference of opinions were the main determinants for this result to obtain. This conclusion is expected to be significantly altered once the setup of Section 3 is considered. In particular, the properties of the disagreement g in (5), which have been highlighted above, are anticipated to impact the way prices respond to current observable shocks: During times when economic conditions deteriorate, both the disagreement along with its persistence increase. As a result, prices are expected to respond more sluggishly during bad news episodes. Yet, when the forecasts of Agent A head towards the bottom state, g notably loses of its persistence and prices more rapidly react to current observable shocks. That is, as the likelihood of recovery increases, prices incorporate current shocks in a more responsive fashion. In the spirit of Bloom (2009) which concentrates on the very different matter of real investment and unemployment, we perform a crude test of this particular implication and report strong empirical support.

On the other hand, the model of Section 3 also carries implications for derivatives assets and more specifically, for risk reversals. Risk reversals are strategies that involve selling an out-the-money call option and buying an out-the-money put option with symmetric strikes. The main idea for considering this particular strategy is as follows: As the economy starts to provide hints of a depression, agents tend to buy put options in order to hedge themselves against recession risk. As a result, in bad times, a higher probability of a recession is associated with an increase in the level of risk reversals. In other words, in such periods, the demand for puts as a hedge increases. Whereas, during more neutral periods, put and call options roughly offset each other and, thus, risk reversal levels remain low. This way of identifying recession risk is pursued in Farhi, Gabaix, Fraiberger, Ranciere, and Verdelhan (2009) in the particular case of exchange rates. We find that our model delivers similar implications for stock returns and interest rates.

4.1 Implications for Price Responses

A notable fact regarding equilibria with learning is that agents use the observable information they have at hand in order to infer the dynamics that they cannot observe. As a result, observable shocks not only appear in the observable state variables but in the filtered hidden states as well. Hence, the equilibrium is expected to be very sensitive to observable shocks. In the case of the model exposed in the last section, both agents focus their attention on the dividend shock $\widehat{W}$ in order to learn about dividend growth. Since there is only one observable shock, this provides us with a very clean way of identifying how the observable shock propagates within the equilibrium dynamics of interest. In particular, and most importantly, a decrease in $\widehat{W}$ is expected to have a very different interpretation in the eyes of both agents: For Agent A, a bad shock in $\widehat{W}$ causes the recession to be more likely, whereas, for Agent B, a bad shock is associated with a bearish cycle at most. Therefore, in our model, it is of major interest to pin down how a bad shock in $\widehat{W}$ is accommodated in equilibrium. As an illustration, we provide a simulated price path in Figure 3. Figure 3a depicts the simulated path of $\widehat{f}$ in blue and $\widehat{f}^B$ in red. Around time $t = 4$, a perceived recession occurs, a situation which we may refer to as DDD in the next subsection (see Subsection 4.1.1). At that point, Agent A's forecasts head to the bottom state f^d while Agent B's expectations reaches a bottom cycle. In Figure 3b, we observe strong price pressures around that very date. As explained earlier in Subsection 2.2, the counterintuitive increase in the stock price following a series of bad shocks is due to time-additive constant relative risk aversion preferences and could be reversed using recursive preferences[17]. These significant moves in price arise

[17] See Section H for further details.

because of the large disagreement during bad times. This is confirmed in Figure 3c where we observe that volume dries up during market recession episodes. This is an interesting implication of our model: Heterogenous beliefs are usually associated with individuals taking bets against each other. One would consistently expect our model to exacerbate the extent of these bets: The opposite situation however turns out to happen. In periods characterized by a long streak of bad news, the disagreement between Agent A and Agent B widens due to Agent A's extreme beliefs. Still, because both individuals' expectations drop significantly below their unconditional mean (f_∞ and $\bar{f}$, respectively), they both anticipate an imminent dividend growth increase. As a result, both agents are willing to post orders in the same direction and trading volume collapses. We now turn to rigorously pin down that phenomenon.

[Insert Figure 3 about Here]

To that purpose, we proceed using impulse response functions which are widely used in Macroeconomics. In the present continuous-time setup, this purpose is achieved through stochastic calculus of variations, namely Malliavin calculus. In particular, the Malliavin derivative measures the change in a path-dependent function implied by a small change in the initial value of the underlying Brownian motion. Put differently, it provides the impulse response function following a shock in the initial values. Applications of Malliavin calculus to Financial Economics have been used in Detemple and Zapatero (1991); Detemple, Garcia, and Rindisbacher (2003). In particular, Berrada (2006) and Dumas, Kurshev, and Uppal (2009) are the first contributions, to the best of our knowledge, that recognize the importance of Malliavin calculus in analyzing equilibria with learning. In order to gather some more concrete intuition, in Subsection 4.1.2 we provide an extensive discussion of the empirical counterpart to a Brownian shock in the context of our model.

Proposition 9 provides an insightful decomposition of the volatility in terms of short-term and long-term components.

Proposition 9. *The volatility can be decomposed as*

$$
\begin{aligned}
\sigma_t &= \theta_t + \frac{\phi_t}{\xi_t S_t} \\
&= \sigma_\delta + \frac{1-\alpha}{\xi_t S_t}\mathbb{E}\left[\int_t^\infty \xi_s \delta_s \int_t^s \mathscr{D}_t \widehat{f}_u du ds \middle| \mathscr{F}_t\right] \\
&+ \frac{1}{\xi_t S_t}\mathbb{E}\left[\int_t^\infty \xi_s \delta_s \left(\omega\left(\eta_t\right) - \omega\left(\eta_s\right)\right) \frac{\mathscr{D}_t \eta_t}{\eta_t} ds \middle| \mathscr{F}_t\right] \\
&- \frac{1}{\xi_t S_t}\mathbb{E}\left[\int_t^\infty \xi_s \delta_s \left(1 - \omega\left(\eta_s\right)\right) \left(\frac{1}{\sigma_\delta}\int_t^s \mathscr{D}_t g_u d\widehat{W}_u + \frac{1}{\sigma_\delta^2}\int_t^s g_u \mathscr{D}_t g_u du\right) ds \middle| \mathscr{F}_t\right]
\end{aligned}
$$

because

$$
\begin{aligned}
\phi_t &= \mathbb{E}\left[\int_t^\infty \xi_s \delta_s \frac{\mathscr{D}_t \xi_t}{\xi_t} ds \middle| \mathscr{F}_t\right] + \mathbb{E}\left[\int_t^\infty \xi_s \delta_s \frac{\mathscr{D}_t \delta_t}{\delta_t} ds \middle| \mathscr{F}_t\right] \\
&+ (1-\alpha)\,\mathbb{E}\left[\int_t^\infty \xi_s \delta_s \int_t^s \mathscr{D}_t \widehat{f}_u du ds \middle| \mathscr{F}_t\right] \\
&+ \mathbb{E}\left[\int_t^\infty \xi_s \delta_s \left(\omega\left(\eta_t\right) - \omega\left(\eta_s\right)\right) \frac{\mathscr{D}_t \eta_t}{\eta_t} ds \middle| \mathscr{F}_t\right] \\
&- \mathbb{E}\left[\int_t^\infty \xi_s \delta_s \left(1 - \omega\left(\eta_s\right)\right) \left(\frac{1}{\sigma_\delta}\int_t^s \mathscr{D}_t g_u d\widehat{W}_u + \frac{1}{\sigma_\delta^2}\int_t^s g_u \mathscr{D}_t g_u du\right) ds \middle| \mathscr{F}_t\right].
\end{aligned} \tag{10}
$$

Proof. See Appendix F. Q.E.D.

The four terms appearing in the volatility decomposition all have very intuitive interpretation. Following Dumas, Kurshev, and Uppal (2009), the first term pertains to the usual myopic portfolio that intends to extract the immediate Sharpe ratio. The other terms all pertain to intertemporal hedges which relate to anticipation of future outcomes. These terms, thus, play a crucial role in the present model. The second term captures hedging against future shocks in $\widehat{f}$. That is, it reflects Agent A trying to hedge the potential catastrophic drops in dividend growth. The third term represents a hedge against future movement in the equilibrium distribution of consumption. This also plays an important role in the analysis as it captures Agent A's anticipation of a contraction of the fraction of trading counterparties in the future. Finally, the last term acts as a hedge against the product of Agent A's future consumption share $\omega\left(\eta_T\right)$ with the future shocks to g_u, $u \in [t, T]$.

Proposition 10 provides the expressions for the Malliavin derivatives of the price of a single-dividend and a single-consumption claim respectively.

Proposition 10. *The expectation at time $t \leq u$ of the Malliavin derivative $\mathscr{D}_t S_u^T$ of the price at date u of a single-maturity unit of dividend to be paid at time $T > u$*

satisfies

$$\mathbb{E}\left[\left.\mathscr{D}_t S_u^T\right|\mathscr{F}_t\right] = \frac{\mathscr{D}_t\delta_t}{\delta_t}\mathbb{E}\left[\left.S_u^T\right|\mathscr{F}_t\right] + \mathbb{E}\left[\left.\frac{\xi_T}{\xi_u}\delta_T\left(\omega\left(\eta_u\right)-\omega\left(\eta_T\right)\right)\frac{\mathscr{D}_t\eta_t}{\eta_t}\right|\mathscr{F}_t\right] \tag{11}$$
$$+\mathbb{E}\left[\left.\frac{\xi_T}{\xi_u}\delta_T\left((1-\alpha)\int_t^T \mathscr{D}_t\widehat{f}_s ds + \alpha\int_t^u \mathscr{D}_t\widehat{f}_s ds\right)\right|\mathscr{F}_t\right]$$
$$-\mathbb{E}\left[\left.\frac{\xi_T}{\xi_u}\delta_T\left(\frac{1}{\sigma_\delta}\int_u^T \mathscr{D}_t g_s d\widehat{W}_s + \frac{1}{\sigma_\delta^2}\int_u^T g_s\mathscr{D}_t g_s ds\right)\right|\mathscr{F}_t\right]$$
$$-\mathbb{E}\left[\left.\frac{\xi_T}{\xi_u}\delta_T\omega\left(\eta_u\right)\left(\frac{1}{\sigma_\delta}\int_t^u \mathscr{D}_t g_s d\widehat{W}_s + \frac{1}{\sigma_\delta^2}\int_t^u g_s\mathscr{D}_t g_s ds\right)\right|\mathscr{F}_t\right]$$
$$+\mathbb{E}\left[\left.\frac{\xi_T}{\xi_u}\delta_T\omega\left(\eta_T\right)\left(\frac{1}{\sigma_\delta}\int_t^T \mathscr{D}_t g_s d\widehat{W}_s + \frac{1}{\sigma_\delta^2}\int_t^T g_s\mathscr{D}_t g_s ds\right)\right|\mathscr{F}_t\right].$$

The expectation at time $t \leq u$ of the Malliavin derivative $\mathscr{D}_t F_u^T$ of the price at date u of a single-maturity unit of consumption to be consumed at date $T > u$ satisfies

$$\mathbb{E}\left[\left.\mathscr{D}_t F_u^T\right|\mathscr{F}_t\right] = \left(\frac{\mathscr{D}_t\delta_t}{\delta_t} - \frac{1}{\alpha}\frac{\mathscr{D}_t\eta_t}{\eta_t}\right)\mathbb{E}\left[\left.F_u^T\right|\mathscr{F}_t\right] \tag{12}$$
$$+\mathbb{E}\left[\left.\frac{\xi_T}{\xi_u}c_T\left((1-\alpha)\int_t^T \mathscr{D}_t\widehat{f}_s ds + \alpha\int_t^u \mathscr{D}_t\widehat{f}_s ds\right)\right|\mathscr{F}_t\right]$$
$$+\mathbb{E}\left[\left.\frac{\xi_T}{\xi_u}c_T\left(\begin{array}{c}\frac{1-\alpha}{\alpha}\left(\frac{1}{\sigma_\delta}\int_t^T \mathscr{D}_t g_s d\widehat{W}_s + \frac{1}{\sigma_\delta^2}\int_t^T g_s\mathscr{D}_t g_s ds\right)\\ +\frac{1}{\sigma_\delta}\int_t^u \mathscr{D}_t g_s d\widehat{W}_s + \frac{1}{\sigma_\delta^2}\int_t^u g_s\mathscr{D}_t g_s ds\end{array}\right)\right|\mathscr{F}_t\right]$$
$$-\mathbb{E}\left[\left.\frac{\xi_T}{\xi_u}c_T\omega\left(\eta_u\right)\left(\frac{g_t}{\sigma_\delta} + \frac{1}{\sigma_\delta}\int_t^u \mathscr{D}_t g_s d\widehat{W}_s + \frac{1}{\sigma_\delta^2}\int_t^u g_s\mathscr{D}_t g_s ds\right)\right|\mathscr{F}_t\right]$$
$$+\frac{\alpha-1}{\alpha}\mathbb{E}\left[\left.\begin{array}{c}\frac{\xi_T}{\xi_u}c_T\omega\left(\eta_T\right)\\ \left(\frac{g_t}{\sigma_\delta} + \frac{1}{\sigma_\delta}\int_t^T \mathscr{D}_t g_s d\widehat{W} g_s + \frac{1}{\sigma_\delta^2}\int_t^T g_s\mathscr{D}_t g_s ds\right)\end{array}\right|\mathscr{F}_t\right].$$

Proof. See Appendix G. Q.E.D.

We emphasize that, in Proposition 10, *expected* impulse response functions are derived. This makes an important difference with what is usually done in Macroeconomics: Impulse response functions are usually computed *assuming* that no further random noise occurs after the function has been perturbed. That is, the impulse response function follows a deterministic path. In the case of Proposition 10, future randomness is accounted for. As a consequence, the impulse response functions $\mathscr{D}_t S_u^T$ and $\mathscr{D}_t F_u^T$ are stochastic. We thus take expectations and obtain the average path of the impulse response functions. To our knowledge, it is the first paper that analyzes the future impact of a shock in today's Brownian motion.

4.1.1 Deciphering a Bearish Cycle

In the remainder, we shall case-study the different phases of a depressing growth episode and consider five cases of interest. Since we wish to emphasize the impact of the fear of recessions, we shall focus our attention on a bearish cycle in the Ornstein-Uehlenbeck model of Agent B. In particular, we identify the following five phases: *i)* First, a slight decline after a bullish period. At this point in the cycle, the economy slowly starts to exhibit signs of a decline after a series of bad shocks. On the one hand, Agent A attributes equal weights to the *steady* and *disaster* states, a situation which has been described as the most uncertain in Section 2. On the other hand, Agent B, whose forecasts reach their long-term mean $\bar{f}$, also attributes equal probabilities to an upward and a downward move. We shall refer to this situation as UD. *ii)* In the second phase of the bearish cycle (DD), another series of bad shocks provides Agent A with a strong hint that the economy is entering a recession. Agent A's expectations have dropped dramatically, while Agent B is adjusting his beliefs towards a downward cycle. Cases UD and DD capture the difference in adjustment speed of Agent A's and Agent B's expectations. In particular, while Agent B's beliefs have almost not changed, Agent A made a significant reassessment of his beliefs. iii) Another series of bad shocks tilts Agent B's filter deep into the bearish cycle while Agent A's model reaches the bottom state. We shall denote this situation by DDU. iv) A final series of bad shocks definitely drifts Agent B's expectations towards the bottom of the bearish cycle while Agent A's expectations still head towards the *disaster* state. This phase, which we shall refer to as DDD, characterizes the worst episode of the bearish cycle. *v)* In the fifth and last phase, a series of good shocks kicks back in. On the one hand, Agent A, who attributes a high probability to recovery, significantly readjusts his beliefs toward a higher level. On the other hand, Agent B's expectations slowly revert to the long-term mean, a phase which we shall call DU. This phase identifies the point at which both Agents' expectations meet again. This occurs around $\widehat{f} = -0.03$, the point at which Agent A is the least assured as to the dividend growth's next step. Hence, although agents do not disagree about the dividend growth at that point, Agent A is uncertain as to future dividend moves while Agent B anticipates an upward shift. The five phases are illustrated in Figure 4 and the associated calibrations are exposed in Table 1. The calibration for the parameters of the model has been adapted from Brennan and Xia (2001) and Dumas, Kurshev, and Uppal (2009). The parameters of the Markov chain have been chosen such that the unconditional average f_∞ matches the long-term mean $\bar{f}$ of the Ornstein-Uehlenbeck process while incorporating a large negative value in the *disaster* state. This calibration is exposed in Table 2. We briefly comment on the type of conditional moments that this calibration reproduces over a bearish cycle[18]: During phase UD, when the economy starts to exhibit signs of a recession, both agents are most uncertain whether the recent drop in dividend growth provides evidence of bad economic outlooks or that these bad shocks are simply transitory. The Sharpe ratio is very low and is located around -0.74. Consistently, the volatility is very weak (around 0.006) and so is the risk premium (around -0.5%). As we move to phase DD, Agent A significantly adjusts his expectations while Agent B remains agnostic regarding future market moves. The latter situation commands a higher market price of risk and the Sharpe ratio along with the risk premium are, in turn, tremendously increased (around 1.58 and 16%, respectively). During phases DDU and DDD, further bad news tilt both agents' expectations towards lower levels of their priors and both Sharpe ratio and risk premium decrease but remain of significant magnitudes of roughly 0.9 and 5%, respectively. During the last phase (DU), the economy takes off again and the Sharpe ratio along with the risk premium severely drop (-0.9 and -1.4%, respectively) while volatility remains high (around 0.16) which reflects Agent B expecting an upward shift while Agent A remains uncertain with respect to future outcomes. These conditional moments reproduce a main feature of the data: The Sharpe ratio and the risk premium are countercyclical, as shown by Fama and French (1989) and Ferson and Harvey (1991). Indeed, risk premia rise in periods of bad economic moves. In order to complete the description of relevant economic variables, we separately expose the behavior of the riskfree rate. The reason is that we simply found unrealistic orders of magnitude using a share of consumption $\omega = \frac{1}{2}$: When both agents represent an identical fraction of the population, the terms of trade on the bond need to adjust beyond realistic levels in order to accommodate both agents' strong desire to lend during recessions. We prefer to set the share of consumption to $\omega = \frac{9}{10}$ for the sole purpose of this paragraph discussion as it does not entail any loss of generality[19] nor does it harm the main results of our paper. Doing so, the riskfree rate lies around 1.3% during the first decline: There is no strong pressures either to borrow or to lend among agents. During the second episode, the riskfree rate significantly increases to 12.8% as Agent A is willing to borrow. Whereas, during the worst phases of the bearish cycle, the riskfree rate dramatically drops to roughly -14% in order to reflect that both agents ardently wish to lend. During episode DU, outlooks appear to be brighter to both agents and the urge to lend decreases in such a way that the riskfree rate reaches -8.5%. This calibration exercise intuitively illustrates agents' trading

[18]The magnitudes reported in the main text are obtained for a share of consumption $\omega = \frac{1}{2}$, except for the riskfree rate for which we set $\omega = \frac{9}{10}$ for the reasons mentioned in the same paragraph.

[19]The pattern of the moves in the riskfree rate roughly remains the same.

strategies over a bad economic period. Also, as shown in Wachter (2011), recursive utility could be used in order to smooth the behavior of the riskfree rate.

[Insert Tables 1 and 2 and Figure 4 about Here.]

In order to provide a reference for each of the cases described above, we first expose the decomposition of the volatility as per Proposition 9 along with the price and wealth sensitivities as per Proposition 10 and volume as defined in (9) for the case where $\widehat{f} = \widehat{f}^B = \bar{f}$, a situation of status quo.

[Insert Figure 5 about Here]

An immediate observation follows by comparing Figure 5a with any of the price responses in Figure 2 of Section 2: The order of magnitudes is now always above, even for the worst case of the Benchmark model. That is, differences of beliefs induce a larger price impact irrelevant of the state of the economy. This is due to the effect of g adding up to the effect of $\widehat{f}$. This is confirmed by the decomposition of the volatility. In particular, it may be observed that g mainly drives the behavior of the volatility as compared to any other term in the volatility decomposition of Proposition 9. Second, and most importantly, the impulse response does not immediately come back to a steady-state level after we perturb today's Brownian motion. More precisely, the stock experiences an immediate drop[20] of 15% which is followed by a sequence of further drops of larger magnitude. This means, in turn, that a shock today has a persistent impact in both prices and wealth. This persistence is produced by the disagreement between Agent A and Agent B. Figure 5a shows that the market initially under-reacts to the current shock and then over-reacts. Moreover, it will prove very important for the analysis to emphasize that *whenever* $\widehat{f}$ *drops below* -0.03 *and* $\widehat{f}^B$ *drops below* $\bar{f}$*, Agent A and Agent B respectively anticipate a future increase in the expected growth rate.* This fact turns out to drive most of the results. In this particular case, it means that Agent B attributes equal probabilities to a dividend growth decrease or increase while Agent A anticipates a decrease. Given Agent B's smooth forecasts, Agent A expects an upward or downward move in Agent B's expectation to only slightly tilt his asset demand toward a buy or a sell offer. As a result, volume remains quite large because none of the agents are expected to have

[20] We emphasize that the percentage change that we shall refer to during the analysis should be compared to a 100% shock in the Brownian. In this particular case, this means that for a change of 1% in the Brownian, the stock price will experience a 0.15% drop. Given the present continuous-time setup, the confidence interval for the Brownian motion change is roughly -10% to 10%.

a strong asset demand in a near future. Yet, as shown in Figure 5c, because Agent A is more inclined to sell, volume is larger for lower value of Agent A's consumption share ω. As expected, the volatility in Figure 5d is maximized when the volume is the largest. With this case at hand, we shall now proceed to describe the different phases of a bearish cycle as exposed above.

i) Case UD is characterized by the peculiar situation in which both agents attribute equal weights to an increase or a decrease in the expected growth rate of dividends. That is, both agents are inclined to speculate. Therefore, a move in the Brownian motion provides a precious hint as to the direction of f. As a consequence, both prices and wealth are very sensitive to a move in the observed shock $\widehat{W}$ as illustrated in Figures 6a and 6b. From these figures, we observe that a shock today has a tremendously persistent effect on future agents' positions and on the price. This is, in particular, striking for the wealth: A shock today induces a permanent change in Agent A's position. This is reminiscent of the fact that Agent A's beliefs move very fast away from the current state. This persistence is confirmed in Figure 6d where the volatility is seen to be almost entirely driven by the disagreement g. Unlike Veronesi (2000), the present situation, which has been described as the most uncertain in Section 2, does not coincide with the highest volatility case. The reason for this result is apparent in Figure 6c in which the volume is very high for a small range of consumption shares. Specifically, volume is only expected to be high when both populations have equal weights. This is very intuitive given that both population of agents are uncertain as to the direction of dividend growth: A small move in either direction could dramatically alter the situation. A move in either direction will cause both agents to have the same demand and to accordingly kill the volume. This fact shall create price pressures whenever ω slightly shifts away from $\frac{1}{2}$.

[Insert Figure 6 about Here]

ii) Case DD captures the difference of adjustment speed between Agent A's and Agent B's expectations: While price and wealth sensitivities remain almost unchanged (see Figures 7a and 7b) as compared to Figures 6a and 6b, volume and volatility are significantly altered. Because of the previously mentioned persistence in price and wealth sensitivities, the volatility level is still driven by the disagreement g. However, now the shape of the volatility is mainly controlled by the third decomposition term in Proposition 9. This term aims to hedge the future changes in the consumption share. Unlike Dumas, Kurshev, and Uppal (2009), this term starts playing a role in the volatility behavior. It stems from a large disagreement between the two agents, a situation that would hardly ever occur in the latter reference. In particular, in Dumas, Kurshev, and Uppal (2009), g oscillates around 0 in a non persistent fashion and, thus, is not amenable to large differences of opinion. In the present case, this leads to a vast surge in volatility. Agent B is still uncertain as to the direction of dividends, while Agent A, who attributes a high probability to recovery, anticipates a significant dividend increase in the near future and is accordingly strongly willing to sell. As seen from Figure 7c, this fact decreases the level of the overall volume. Yet, given that the distribution of future consumption is now a relevant concern, as seen from the volatility decomposition in Figure 7d, the volume spreads out around $\frac{1}{2}$. Finally, notice that Agent A being strongly willing to sell, volume is the highest for lower values of ω.

[Insert Figure 7 about Here]

iii) Case DDU depicts the situation in which both agents are deep into the bearish phase and are accordingly expecting an imminent upward shift in dividend growth. From Figure 8c, it is apparent that the overall volume has experienced a further decrease. As a result, the wealth sensitivity is decreased (see Figure 8b). This is due to agents' willingness to adjust their portfolio holdings in the same direction. We emphasize this particular implication of our model: Although the deterioration of economic conditions further magnifies the difference of opinions between the two agents, agents both expect dividend growth to increase in a very near future. As a result, instead of taking large bets against each other motivated by their strongly different views, they are both willing to rebalance in the same direction. Also, low volume contributes to strong pressures on prices. As confirmed by Figure 8a, a shock in the Brownian today induces a tremendous downward move in the stock price. Also apparent in this figure is the fact that persistence has almost disappeared. This is better seen in the volatility decomposition of Figure 8d where the shape starts to be driven by future moves in the consumption share. The disagreement g now only plays a role for relatively low values of ω. This aspect is reminiscent of the behavior of g which we discussed in Section 3: the disagreement g looses its persistence once the expected growth rate wanders closer to f^d.

[Insert Figure 8 about Here]

iv) Case DDD characterizes the worst episode of the cycle. At that point, both agents have reached a drastically low level of their priors and anticipate an upward shift at any time. Consequently, results pertaining to volume and wealth are preserved. Most importantly, all the remaining persistence has now washed out and the shape of the volatility is entirely driven by future consumption moves (see Figure 9d). This should be considered as the result of both agents attributing a null probability to any further bad shock. Since it is almost impossible for the situation to further deteriorate, prices are much less sensitive to bad shocks and price pressure widely attenuate (see Figure 9a). These facts are consistent with the disagreement g having lost all of its persistence at this point of the cycle.

[Insert Figure 9 about Here]

v) The last case of interest corresponds to the situation where both agents have made a significant reassessments of their priors and agree about the current level of dividend growth. Despite of the latter agreement, Agent A is now most uncertain about dividend's next move while Agent B anticipates a robust increase in f. As a first implication, future moves in ω have now become irrelevant in the volatility decomposition (see Figure 10d). From this figure, it is observed that persistence start to kick back in. This is further confirmed in Figure 10a where the price exhibits lagged responses. Most interesting is the following: On the one hand, Agent B who thinks that dividend will experience a stable increase is inclined to sell. On the other hand, Agent B knows that Agent A adjusts his expectation very fast and, as a result, anticipates a vast upward shift in $\widehat{f}$ if a positive shock occurs. Since Agent A attributes high probability to recovery, Agent B expects Agent A to be willing to aggressively sell in a near future. Consequently, markets are vastly frozen for any ω. Since volume is shown to be very low during a phase of recovery, price pressures arise again. From Figure 10a, a shock today is amenable to a large and persistent variation in prices.

[Insert Figure 10 about Here]

Before turning to the empirical support for our model along with another of its implications, we wish to briefly comment on market selection[21]concerns in the

[21]See for instance Dumas, Kurshev, and Uppal (2009), Berrada (2006, 2009), Bhamra and Uppal (2010), Kogan, Ross, Wang, and Westerfield (2006) and Rogers (2010) for detailed results on the topic.

present setting. In particular, given the behavior of Agent A, it is natural to ask oneself in which conditions Agent A survives or dies. Assuming that Agent B has the correct beliefs, Agent A appears to survive for a surprisingly long time period[22]. Hence, our analysis is not subject to market selection concerns which would arise, were Agent A to rapidly disappear after a sequence of bad shocks.

4.1.2 Empirical Support

We now proceed to test whether the model prediction, which we just exposed, achieves some empirical support. We do not intend to perform a sophisticated econometric study in the present article, and we leave such a task for future research. Instead, our goal is merely to roughly investigate whether our model finds some appeal in the data. Also, despite the relatively detailed implications described above, we restrict, to that purpose, our attention to the following more general claim: *Prices should be more responsive to current shocks in bad times than in good times.* In a recent study, Bloom (2009) investigates how uncertainty shocks propagate within a production economy with a particular focus on second moments effects for real investment and employment. In the following, we attempt to gauge the first moment impact of news shocks on prices. Our analysis may be partly seen as the financial counterpart to the latter reference. Also, as in Lustig and Verdelhan (2010), we condition our empirical investigation on different phases of the business cycle, consistent with the predictions of our model.

We articulate our empirical test in two main steps: *a)* We first need to specify the empirical counterpart to the observable shock $\widehat{W}$ of our model. Since the observable shock is represented by the innovation in the dividend dynamics, the first experiment that comes to mind is the following: Using the US GDP growth time series as a proxy for the aggregate output δ along with the forecasts of US GDP growth as a proxy for the filter $\widehat{f}$, one could easily back out the observable Brownian motion $\widehat{W}$ without having to resort to some complicated hidden state filtering techniques. This exercise is, however, unfortunately compromised by the frequency of GDP growth which is only available quarterly. We circumvent this difficulty by appealing to another feature of our model: the Markov chain of Agent A. More precisely, Agent A filters out information by looking at the observable shock $\widehat{W}$. In that matter, we stressed earlier that Agent A becomes concerned by recessions only when economic conditions deteriorate and that the filtered Markov chain precisely reflect such a behavior. Taking a more real to life standpoint, were Agent A a man of flesh and blood, he would indeed be very much prone to actively search for terms that best

[22] More precisely, Agent A survives over a period which extends way beyond one hundred years.

reflect his current worries. Thence, as in Lustig and Verdelhan (2010), this increasing attention to recession risk seem to be conveniently captured by Google search volume changes for terms related to financial distress. This measure not only allows to bypass the empirical difficulty mentioned above, but it also achieves our premise that recession risk becomes of an increasing concern during bad times. Thus, it appears both reasonable and appealing to extract the observable Brownian motion by considering a measure of Google search volume changes. In particular, we consider weekly Google search volume changes from 2004 to 2010 on the distressed terms: financial crisis, crisis, crises, recession, and bankruptcy. This list of terms may not appear to be very exhaustive. As it turns out, these are however the few terms that most accurately approximate our notion of perceived recession. We then build a demeaned value weighted search volume index: Changes in the index are recorded and standardized (variance equal to 1 week) in such a way that this time series can be interpreted as changes in a standard Brownian motion. We call this time series x_{1t} for $t \in [0, 6]$ and consider it as a proxy for the observable change in the Brownian motion $\widehat{W}$.

Second, *b)* we need to empirically distinguish between periods of good and bad times. Within the theoretical analysis framework, we do so by considering the cycles in the dividend growth as filtered by the agent with moderate views. This points to the following empirical counterpart: One can consider periods during which forecasts of the GDP growth lie below or above a certain threshold. Specifically, we define good times (bad times) as: (*i*), the periods in which the GDP growth forecasts are above (below) their sample average, (*ii*), the periods in which the GDP growth is above (below) its sample average. Both of these empirical proxies appear to be reasonably close to our theoretical study[23]. Both GDP growths and growth forecasts are downloaded from the Federal Reserve Bank of Philadelphia website.

With these two empirical proxies at hand, we perform the following regression[24]

$$r_{t+k} = \alpha + \beta_1 x_{1t} + \beta_2 x_{2t} + \epsilon_t, \quad 0 \leq k \leq \frac{1}{2}$$

where r denotes the S&P 500 weekly return downloaded from Datastream, k is the lag, x_1 represents the change in the search volume index, and $x_{2t} \equiv x_{1t}\mathbf{1}_{\{z_t < \bar{z}\}}$. The time series z either stands for the GDP growth forecast or the GDP growth itself. Finally, $\bar{z}$ is the empirical average of z between 1969 to 2010. Importantly, we

[23]We could alternatively use NBER recession dates or OECD economic turning points for the US, as in Lustig and Verdelhan (2010). However, the NBER has identified only 32 cycles since 1854, which is too low a number for our empirical study to make sense.

[24]We use the Newey-West estimator for the variance-covariance matrix of estimates.

attribute the following interpretations to the coefficients of the above regression: On the one hand, β_1 represents the S&P 500 price response to a change in the Brownian motion, conditional on being in a good times state. On the other hand, $\beta_1 + \beta_2$ represents the S&P 500 price response to a change in the observable Brownian shock $\widehat{W}$, conditional on being in a bad times state.

[Insert Figures 11 and 12 and Tables 3 and 4 about Here]

In Figure 11, we plot the evolution of the impulse response in good times (blue) and in bad times (red) when good times are defined using the GDP growth forecasts. In Figure 12, we proceed to the same exercise only using GDP growth as a criterion for defining good times and bad times. From both figures, we observe the following striking fact: In both cases, the magnitude of the impulse response is significantly larger in bad times, thus confirming the prediction of our model. Furthermore, Tables 3 and 4 provide the contemporaneous ($k = 0$) regression outputs for the cases (i) and (ii), respectively. First, β_2 is, in both cases, significant and negative. In other words, a positive volume change on bad news implies significantly negative returns in bad times and the price response to a current shock is therefore more important in bad times. Since β_1 is significant in case (ii) only, a positive volume change on bad news implies significantly negative returns in good times only in case (ii). In summary, performing a crude empirical investigation, we are able to show that the prediction of our model finds support in the data. Although we wholeheartedly admit that the latter experiment leaves room for a much more thorough empirical study, we still believe that our empirical measure hints to some relevant support for our model. This also delivers additional testable implications which are left for further research.

4.2 Implications for Risk Reversals

As mentioned above, risk reversals provide an intuitive way to measure investor's desire to hedge against recession risk. During favorable economic episodes, this type of risk tend to be neglected and, as a result, risk reversals levels remain low. As a streak of bad news kicks in, recession risk becomes more relevant and, in an attempt to hedge themselves, investors buy put options. As a result, the risk reversals level rises. Given Agent A's view on possible future outcomes, it is of interest to use the equilibrium stock price dynamics of our model in order to evaluate risk reversals. Doing so, we can assess how risk reversals levels relate to the main equilibrium quantities of our model. In particular, we look at their relation with interest rates and stock returns. In that respect, we can anticipate the following mechanisms: On the one hand, when economic outlooks worsen, Agent A is willing to liquidate part of his holdings in the stock and to lend the proceeds of the sale to Agent B. As economic conditions further deteriorate, Agent B becomes more and more reluctant to borrow and Agent A more and more willing to lend. As a result, the terms of trade need to adjust in order to accommodate both agents' positions. In doing so, the equilibrium riskfree rate naturally has to decrease, which implies, in turn, a negative relation with risk reversals levels. On the other hand, a sustained sequence of bad news intuitively induces negative stock price returns and, thus, a negative relation with risk reversals levels as well.

In order to confirm this intuition and obtain the model implied properties for the risk reversals, we simulate the economy at a daily frequency over a 20 years horizon. Using these time series, we price $25-\Delta$ call and put options with maturity 1 month. These are out-of-the money options with strike prices K_c and K_p, respectively. The strikes K_c and K_p are simply given by the Black-Scholes formula and satisfy

$$K_c = Se^{(\bar{r}-\bar{y}+\frac{1}{2}\bar{\sigma}^2)\tau-\bar{\sigma}\sqrt{\tau}\Phi^{-1}(0.25e^{\bar{y}\tau})}$$
$$K_p = Se^{(\bar{r}-\bar{y}+\frac{1}{2}\bar{\sigma}^2)\tau+\bar{\sigma}\sqrt{\tau}\Phi^{-1}(0.25e^{\bar{y}\tau})}$$

where S represents the current stock price, $\bar{r}$ the average risk free rate, $\bar{y}$ the average dividend yield, $\bar{\sigma}$ the average stock return volatility, $\tau = \frac{1}{12}$ the time to maturity, and Φ the cumulative Gaussian distribution function. Moreover, the risk reversal RR_t at time t is defined by

$$RR_t = p_t - c_t$$

where p_t and c_t are the put and call option prices at time t, respectively. Our approach is then inspired from Farhi, Gabaix, Fraiberger, Ranciere, and Verdelhan (2009). The latter contribution empirically investigates the implications of risk reversals for the exchange rate (FX) market. In particular, they show that risk reversals increase with both the interest rates and the currency depreciation. We now consider similar implications associated with stock returns: We run the following regressions.

$$RR_t = \alpha + \beta r_t^f + \epsilon_t$$

and

$$RR_t = \alpha + \beta r_t^S + \epsilon_t$$

where r_t^f denotes the model implied risk free rate at time t and r_t^S is the model implied stock return at time t.

The resulting estimates are exposed in Tables 5 and 6, respectively. We provide the parameter estimates along with their associated t-statistics, and P-values.

[Insert Tables 5 and 6 about Here]

As expected and consistent with the above intuition, both linear fits demonstrate that the slope is negative and significant. This suggests that the risk reversal decreases with both the risk free rate and stock returns. These estimates not only show that the implications of our model are intuitively appealing, but they also imply, as suggested and investigated by Farhi, Gabaix, Fraiberger, Ranciere, and Verdelhan (2009), that risk reversals are potentially well-suited in order to gauge the importance of recession risk in the data.

5 Conclusion

We build a continuous-time pure exchange economy in which agents estimate strictly different models for the fundamental. In particular, one agent fears that substantial recessions in expected dividend growth might occur while the other agent has smoother beliefs. We show that such a difference of models is amenable to a large disagreement during bad times. Specifically, disagreement among agents exhibits a rich pattern during such periods: At the beginning of a bearish phase, the agent who fears potential recessions in output growth significantly reassesses the probability of a recession upwards. The other agent who holds smoother and mostly more persistent beliefs hardly updates his expectations. This translates into initially persistent disagreement among the two agents. Yet, after a long and sustained streak of bad news, both agents expect a recovery to take place in a near future and disagreement gradually loses of its persistence. Contrary to the common wisdom that heterogeneous beliefs induce large bets, this pattern provokes a collapse in trading volume which ultimately leads to strong price pressures as no agents is eager to make the market for the other. Since this mechanism only prevails during bad economic episodes, we condition our analysis on the different phases of a bearish cycle. To that purpose, we base our study on impulse response functions. These are suited to investigate the expected future impact of a shock today on prices, wealth and other functions of interest. Doing so, we are able to cleanly identify phases of price pressures and persistence of shocks. Depending on the economic conditions, the market is shown to first under-react to a current shock and then over-react. More generally, we find that prices react very differently during good and bad times. Using

Google search volume for financial distress terms, we take our model to the data and report that prices indeed react more significantly to news shocks during bad times.

The present model could be extended to an international finance setting to investigate the home bias puzzle. In particular, one could model home investors as being more cautious about foreign shocks. For instance, it appears reasonable to assume that US investors have very different models in mind regarding the evolution of future home and Brazilian dividend growths. The resulting large difference of beliefs is likely to produce heavily tilted portfolios towards home assets.

A Proof of Proposition 4

Define the conditional expectation $V\left(t;f^{i}\right)$ for $i\in\{s,d\}$ as

$$V\left(t;f^{i}\right)\equiv\mathbb{E}\left[e^{-\rho(T-t)}\left(\frac{\delta_T}{\delta_t}\right)^{1-\alpha}\middle|f_t=f^i\right].$$

Consider an infinitesimal interval Δ. Conditional on $f_t=f^i$, the two random variables $\frac{\delta_{t+\Delta}}{\delta_t}$ and $\frac{\delta_T}{\delta_{t+\Delta}}$ are independent. As a result, we can rewrite $V\left(t;f^{i}\right)$ as

$$\begin{aligned}
V\left(t;f^{i}\right)&=\mathbb{E}\left[e^{-\rho\Delta}\left(\frac{\delta_{t+\Delta}}{\delta_t}\right)^{1-\alpha}\times e^{\rho(T-(t+\Delta))}\left(\frac{\delta_T}{\delta_{t+\Delta}}\right)^{1-\alpha}\middle|f_t=f^i\right]\\
&=\mathbb{E}\left[e^{-\rho\Delta}\left(\frac{\delta_{t+\Delta}}{\delta_t}\right)^{1-\alpha}\middle|f_t=f^i\right]\times\mathbb{E}\left[e^{\rho(T-(t+\Delta))}\left(\frac{\delta_T}{\delta_{t+\Delta}}\right)^{1-\alpha}\middle|f_t=f^i\right]\\
&=e^{b_i\Delta}\times\mathbb{E}\left[e^{\rho(T-(t+\Delta))}\left(\frac{\delta_T}{\delta_{t+\Delta}}\right)^{1-\alpha}\middle|f_t=f^i\right]\\
&=e^{b_i\Delta}\times\left[\begin{array}{c}\left(1-\sum_{j\neq i}a_{ij}\Delta\right)\mathbb{E}\left[e^{\rho(T-(t+\Delta))}\left(\frac{\delta_T}{\delta_{t+\Delta}}\right)^{1-\alpha}\middle|f_t=f^i\right]\\+\sum_{j\neq i}a_{ij}\mathbb{E}\left[e^{\rho(T-(t+\Delta))}\left(\frac{\delta_T}{\delta_{t+\Delta}}\right)^{1-\alpha}\middle|f_t=f^j\right]\end{array}\right]\\
&=e^{b_i\Delta}\times\left[\left(1-\sum_{j\neq i}a_{ij}\Delta\right)V\left(t+\Delta;f^i\right)+\sum_{j\neq i}a_{ij}V\left(t+\Delta;f^j\right)\Delta\right]
\end{aligned}$$

where $\{a_{ij}\}$ are elements of the generator matrix Λ and $b_i=-\rho+(1-\gamma)f^i-\frac{1}{2}\sigma_\delta^2\gamma(1-\gamma)$. Using a first order Taylor expansion, we can write $V\left(t+\Delta;f^i\right)=V\left(t;f^i\right)+V'\left(t;f^i\right)\Delta+o(\Delta)$. Substituting, we obtain

$$V\left(t+\Delta;f^i\right)=e^{b_i\Delta}\left(V\left(t;f^i\right)+V'\left(t;f^i\right)\Delta-\sum_{j\neq i}a_{ij}V\left(t;f^i\right)\Delta+\sum_{j\neq i}a_{ij}V\left(t;f^j\right)\Delta\right).$$

Rearranging and dividing by Δ, we get

$$V\left(t+\Delta;f^i\right)\frac{1-e^{b_i\Delta}}{\Delta}=e^{b_i\Delta}\left(V'\left(t;f^i\right)-\sum_{j\neq i}a_{ij}V\left(t;f^i\right)+\sum_{j\neq i}a_{ij}V\left(t;f^j\right)\right).$$

Taking the limit as $\Delta\to0$ and rearranging, we obtain the following ODE

$$V'\left(t;f^i\right)=\left(\sum_{j\neq i}a_{ij}-b_i\right)V\left(t;f^i\right)-\sum_{j\neq i}a_{ij}V\left(t;f^j\right)$$

with boundary condition $V(T; f^i) = 1$. The solution to this ODE, in the two-dimensional case of Section 2, is given by

$$V\left(t; f^i\right) = \frac{e^{\frac{1}{2}\left(B-\sqrt{A}(T-t)\right)}\left(\sqrt{A}+(1-\alpha)f^i+(\alpha-1)f^j-\lambda-\psi\right)}{2\sqrt{A}} + \frac{e^{\frac{1}{2}\left(\sqrt{A}(T-t)+B\right)}\left(\sqrt{A}+(\alpha-1)f^i+(1-\alpha)f^j+\lambda+\psi\right)}{2\sqrt{A}}$$

where

$$A = 2\psi\left((\alpha-1)f^d-\alpha f^s+f^s+\lambda\right)+\left((\alpha-1)\left(f^s-f^d\right)+\lambda\right)^2+\psi^2$$

and

$$B = (t-T)\left((\alpha-1)\left(f^d-\alpha\sigma_\delta^2\right)+\lambda+2\rho+\psi\right)+(\alpha-1)f^s(t-T).$$

We finally use the fact that

$$\frac{S_t^T}{\delta_t} = \pi_t V\left(t; f^s\right)+(1-\pi_t)V\left(t; f^d\right) \tag{13}$$

in order to obtain the price of the single-dividend paying stock. □

B Proof of Proposition 5

We follow the notations in Lipster and Shiryaev (2001) and write the observable process as

$$\frac{\mathrm{d}\delta_t}{\delta_t} = \left(A_0 + A_1 f_t^B\right)\mathrm{d}t + B_1 \mathrm{d}W_t^f + B_2 \mathrm{d}W_t^B$$

and the unobservable process as

$$\mathrm{d}f_t^B = \left(a_0 + a_1 f_t^B\right)\mathrm{d}t + b_1 \mathrm{d}W_t^f + b_2 \mathrm{d}W_t^B.$$

Using the SDEs in (1) and (3), observe that $B \circ B = \sigma_f^\delta$, $b \circ b = \sigma_f^2$ and $b \circ B = 0$. Applying Theorem 12.7 in Lipster and Shiryaev (2001), the dynamics of the filter satisfy

$$\mathrm{d}\widehat{f}_t^B = \left(a_0 + a_1 \widehat{f}_t^B\right)\mathrm{d}t + \left(B \circ B + \gamma A_1^\top\right)\left(B \circ B\right)^{-1}\left(\frac{\mathrm{d}\delta_t}{\delta_t} - \left(A_0 + A_1 \widehat{f}_t^B\right)\mathrm{d}t\right)$$

where the steady-state variance γ solves the algebraic equation

$$a_1\gamma + \gamma a_1^\top + b \circ b - \left(b \circ B + \gamma A_1^\top\right)(B \circ B)^{-1}\left(b \circ B + \gamma A_1^\top\right)^\top = 0.$$

Substituting the coefficients, we obtain (4), the steady-state variance and

$$\mathrm{d}\widehat{W}_t^B = \frac{1}{\sigma_\delta}\left(\frac{\mathrm{d}\delta_t}{\delta_t} - \widehat{f}_t^B \mathrm{d}t\right).$$

□

C Proof of Proposition 6

First, define the two following changes of measure

$$\left.\frac{\mathrm{d}\overline{\mathbb{P}}}{\mathrm{d}\mathbb{P}^A}\right|_{\mathscr{F}_t} \equiv e^{-\frac{1}{2}\int_0^t \left(\frac{j}{\alpha}\frac{g_s}{\sigma_\delta}\right)^2 \mathrm{d}s - \int_0^t \frac{j}{\alpha}\frac{g_s}{\sigma_\delta}\mathrm{d}\widehat{W}_s}$$

and

$$\left.\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\overline{\mathbb{P}}}\right|_{\mathscr{F}_t} \equiv e^{-\frac{1}{2}\int_0^t (1-\alpha)^2\sigma_\delta^2 \mathrm{d}s + \int_0^t (1-\alpha)\sigma_\delta \mathrm{d}\overline{W}(s)}$$

where, by Girsanov's Theorem, $\overline{W}$ is a $\overline{\mathbb{P}}-$Brownian motion satisfying

$$\overline{W}_t = \widehat{W}_t + \int_0^t \frac{j}{\alpha}\frac{g_s}{\sigma_\delta}\mathrm{d}s$$

and $\widetilde{W}$ is a $\widetilde{\mathbb{P}}-$Brownian motion satisfying

$$\widetilde{W}_t = \overline{W}_t - \int_0^t (1-\alpha)\,\sigma_\delta \mathrm{d}t. \tag{14}$$

Substituting the above relationships in Equation (7) leads to

$$F_t^j\left(\widehat{f}_t, g_t\right) = \widetilde{\mathbb{E}}\left[\left.\int_t^\infty e^{\int_t^u X_s^j \mathrm{d}s}\mathrm{d}u\right| \mathscr{F}_t\right] \tag{15}$$

where

$$X_t^j = -\left(\rho + \frac{1}{2}(1-\alpha)\,\alpha\sigma_\delta^2\right) + \frac{1}{2}\frac{j}{\alpha}\left(\frac{j}{\alpha} - 1\right)\frac{g_t^2}{\sigma_\delta^2} - (1-\alpha)\,\frac{j}{\alpha}g_t + (1-\alpha)\,\widehat{f}_t$$

and $\widetilde{\mathbb{E}}[.]$ stands for the expectation operator taken under an equivalent probability measure defined in (14).

Second, in order to figure out the partial differential equation the function F^j has to satisfy, we use the fact that the expectation pertaining to Equation (15) can be rewritten as

$$
\begin{aligned}
F_t^j\left(\widehat{f}_t, g_t\right) &= e^{-\int_0^t X_s^j \mathrm{d}s}\widetilde{\mathbb{E}}\left[\int_t^\infty e^{\int_0^u X_s^j \mathrm{d}s}\mathrm{d}u \middle| \mathscr{F}_t\right] \\
&= e^{-\int_0^t X_s^j \mathrm{d}s}\left(-\int_0^t e^{\int_0^u X_s^j \mathrm{d}s}\mathrm{d}u + \widetilde{\mathbb{E}}\left[\int_0^\infty e^{\int_0^u X_s^j \mathrm{d}s}\mathrm{d}u \middle| \mathscr{F}_t\right]\right) \\
&\equiv e^{-\int_0^t X_s^j \mathrm{d}s}\left(-\int_0^t e^{\int_0^u X_s^j \mathrm{d}s}\mathrm{d}u + \widetilde{M}_t\right)
\end{aligned}
$$

where $\widetilde{M}$ is a $\widetilde{\mathbb{P}}-$Martingale.

Finally, Itô's lemma along with the martingale representation Theorem deliver the desired result.

□

D Proof of Proposition 7

Following Duffie (2010), the functions $\alpha(\tau)$ and $\beta(\tau) \equiv (\beta_1(\tau), \beta_2(\tau))$ solve the following system of Ricatti equations

$$
\begin{aligned}
\beta'(\tau) &= K_1^\top \beta(\tau) + \frac{1}{2}\beta(\tau)^\top H_1 \beta(\tau) \\
\alpha'(\tau) &= K_0^\top \beta(\tau) + \frac{1}{2}\beta(\tau)^\top H_0 \beta(\tau)
\end{aligned}
$$

with boundary conditions $\beta(0) = (1-\alpha, 0)$ and $\alpha(0) = 0$ and where

$$
K_0 = \begin{pmatrix} -\frac{1}{2}\sigma_\delta^2 \\ \kappa \bar{f} \end{pmatrix}, K_1 = \begin{pmatrix} 0 & 1 \\ 0 & -\kappa \end{pmatrix}
$$

and

$$
H_0 = \begin{pmatrix} \sigma_\delta^2 & \gamma \\ \gamma & \left(\frac{\gamma}{\sigma_\delta}\right)^2 \end{pmatrix}, H_1 = 0_2 \otimes 0_2.
$$

The solutions to this system are $\beta_1(\tau) = 1-\alpha$, $\beta_2(\tau) = -\frac{(\alpha-1)e^{-\kappa\tau}(e^{\kappa\tau}-1)}{\kappa}$ and

$$
\begin{aligned}
\alpha(\tau) = & \frac{(\alpha-1)^2\gamma^2 e^{-2\kappa\tau}\left(e^{2\kappa\tau}(2\kappa\tau-3)+4e^{\kappa\tau}-1\right)}{4\kappa^3\sigma_\delta^2} \\
& - \frac{(\alpha-1)e^{-2\kappa\tau}\left(4\kappa\sigma_\delta^2 e^{\kappa\tau}\left(e^{\kappa\tau}(\kappa\tau-1)+1\right)\left(\kappa\bar{f}-\alpha\gamma+\gamma\right)+2\alpha\kappa^3\tau\sigma_\delta^4 e^{2\kappa\tau}\right)}{4\kappa^3\sigma_\delta^2}.
\end{aligned}
$$

□

E Proof of Proposition 8

Consider the following Chebyshev polynomial

$$P^j\left(\widehat{f},g\right)=\sum_{i=0}^{n}\sum_{k=0}^{m}a_{i,k}^{j}T_i\left(\widehat{f}\right)T_k\left(g\right)\approx F^j(\widehat{f},g)$$

where T_i is the Chebyshev polynomial of order i. The polynomial P^j will be referred to as being the numerical approximation of the function $F^j(\widehat{f},g)$ for $j=1,\ldots,\alpha-1$. Following Judd (1998), we mesh the roots of the Chebyshev polynomial of order n with those of the Chebyshev polynomial of order m in order to obtain the interpolation nodes. We then substitute $P^j(\widehat{f},g)$ and its derivatives in PDE (8) and evaluate this expression at the interpolation nodes. Since all the boundaries are so-called absorbing, this approach yields directly a system of $(n+1)\times(m+1)$ equations with $(n+1)\times(m+1)$ unknowns which can be solved in a few minutes. □

F Proof of Proposition 9

In the present Section, we derive the decomposition of the volatility which appears in (10). To that purpose, define the martingale

$$M_t\equiv\xi_tS_t+\int_0^t\xi_s\delta_s\mathrm{d}s=\int_0^\infty\mathbb{E}\left[\xi_s\delta_s\middle|\mathscr{F}_t\right]\mathrm{d}s.$$

Applying Itô's lemma on both sides, we obtain

$$\begin{aligned}\mathrm{d}M_t&=\xi_t\left(\sigma_t-\theta_t\right)\mathrm{d}\widehat{W}_t\\&=\phi_t\mathrm{d}\widehat{W}_t\end{aligned}$$

where θ_t denotes the market price of risk and ϕ_t denotes the integrand in the representation of the martingale M_t as a stochastic integral. It follows, then, that the stock return volatility satisfies

$$\begin{aligned}\sigma_t&=\theta_t+\frac{\phi_t}{\xi_tS_t}\\&=\theta_t+\frac{\phi_t}{\int_t^\infty\mathbb{E}\left[\xi_u\delta_u\middle|\mathscr{F}_t\right]\mathrm{d}u}.\end{aligned}$$

By Clark-Ocone theorem, which allows to identify ϕ_t, we get that ϕ_t satisfies satisfies

$$\phi_t = \int_t^\infty \mathbb{E}\left[\delta_u \mathscr{D}_t \xi_u \middle| \mathscr{F}_t\right] \mathrm{d}u + \int_t^\infty \mathbb{E}\left[\xi_u \mathscr{D}_t \delta_u \middle| \mathscr{F}_t\right] \mathrm{d}u.$$

Using that

$$\frac{\mathscr{D}_t \delta_T}{\delta_T} = \frac{\mathscr{D}_t \delta_t}{\delta_t} + \int_t^T \mathscr{D}_t \widehat{f}_u \mathrm{d}u$$

and

$$\frac{\mathscr{D}_t \eta_T}{\eta_T} = \frac{\mathscr{D}_t \eta_t}{\eta_t} - \frac{1}{\sigma_\delta} \int_t^T \mathscr{D}_t g_u \mathrm{d}\widehat{W}_u - \frac{1}{\sigma_\delta^2} \int_t^T g_u \mathscr{D}_t g_u \mathrm{d}u$$

along with the steps used to derived (11) and (12), we obtain the decomposition in (10) (See Appendix G). Notice that here we use an additional trick: Within each the expectation appearing in Proposition 9, we add and withdraw $\frac{\mathscr{D}_t \xi_t}{\xi_t}$. For instance, this would allow to obtain

$$\mathscr{D}_t S_t^T = S_t^T \frac{\mathscr{D}_t \delta_t}{\delta_t} + \mathbb{E}\left[\frac{\xi_T}{\xi_t} \delta_T \left(\frac{\mathscr{D}_t \xi_T}{\xi_T} - \frac{\mathscr{D}_t \xi_t}{\xi_t}\right) \middle| \mathscr{F}_t\right].$$

We then use that

$$\frac{\mathscr{D}_t \xi_t}{\xi_t} = -\alpha \frac{\mathscr{D}_t \delta_t}{\delta_t} + (1 - \omega(\eta_t)) \frac{\mathscr{D}_t \eta_t}{\eta_t}$$

and

$$\begin{aligned} \frac{\mathscr{D}_t \xi_T}{\xi_T} = \frac{\mathscr{D}_t \xi_t}{\xi_t} - \alpha \int_t^T \mathscr{D}_t \widehat{f}_u \mathrm{d}u + (\omega(\eta_t) - \omega(\eta_T)) \frac{\mathscr{D}\eta}{\eta_t} \\ - (1 - \omega(\eta_T)) \left(\frac{1}{\sigma_\delta} \int_t^T \mathscr{D}_t g_u \mathrm{d}\widehat{W}_u + \frac{1}{\sigma_\delta^2} \int_t^T g_u \mathscr{D}_t g_u \mathrm{d}u\right). \end{aligned}$$

As no closed-form solutions are available for $\mathscr{D}_t \widehat{f}_u$ and $\mathscr{D}_t g_u$, we need to simulate them. To do so, we use the approach in Detemple, Garcia, and Rindisbacher (2005) and observe that both Malliavin derivatives respectively satisfy the following SDEs

$$\mathrm{d}\mathscr{D}_t \widehat{f}_u = \frac{\partial}{\partial \widehat{f}} \mu_f \left(\widehat{f}_u\right) \mathscr{D}_t \widehat{f}_u \mathrm{d}u + \frac{\partial}{\partial \widehat{f}} \sigma_f \left(\widehat{f}_u\right) \mathscr{D}_t \widehat{f}_u \mathrm{d}\widehat{W}_u$$

and

$$\mathrm{d}\mathscr{D}_t g_u = \nabla \mu_g \left(\widehat{f}_u, g_u\right)^\top \begin{pmatrix} \mathscr{D}_t \widehat{f}_u \\ \mathscr{D}_t g_u \end{pmatrix} \mathrm{d}u + \frac{\partial}{\partial \widehat{f}} \sigma_g \left(\widehat{f}_u\right) \mathscr{D}_t \widehat{f}_u \mathrm{d}\widehat{W}_u$$

where μ_f and μ_g denote the drift and σ_f and σ_g denote the diffusion of $\widehat{f}$ and g, respectively, that appear in the SDEs (2) and (5), respectively. We simulate these dynamics using a first order Euler discretization scheme. □

G Proof of Proposition 10

We show how (12) is obtained. We do not reproduce the derivation of (11) as it follows similar steps. Define the price F_u^T at time u of a single-maturity unit of consumption to be consumed at date $T > u$ as

$$F_u^T \equiv \mathbb{E}\left[\frac{\xi_T}{\xi_u} c_T \middle| \mathscr{F}_u\right] = \left(\phi e^{\rho T}\right)^{-\frac{1}{\alpha}} \mathbb{E}\left[\frac{\xi_T^{-\frac{1}{\alpha}+1}}{\xi_u} \middle| \mathscr{F}_u\right]. \tag{16}$$

Taking the Malliavin derivative of (16) at time $t \leq u$, we get

$$\mathscr{D}_t F_u^T = \left(\phi e^{\rho T}\right)^{-\frac{1}{\alpha}} \mathbb{E}\left[-\frac{1}{\xi_u^2} \mathscr{D}_t \xi_u \left(\xi_T^{\frac{\alpha-1}{\alpha}}\right) \middle| \mathscr{F}_u\right] + \left(\phi e^{\rho T}\right)^{-\frac{1}{\alpha}} \mathbb{E}\left[\frac{1}{\xi_u} \frac{\alpha-1}{\alpha} \xi_T^{-\frac{1}{\alpha}} \mathscr{D}_t \xi_T \middle| \mathscr{F}_u\right]. \tag{17}$$

Now, we use that, for any $t \leq s \leq T$,

$$\begin{aligned} \frac{\mathscr{D}_t \xi_s}{\xi_s} &= \frac{\mathscr{D}_t \xi_t}{\xi_t} - \alpha \int_t^s \mathscr{D}_t \widehat{f}_u \mathrm{d}u + \left(\omega\left(\eta_t\right) - \omega\left(\eta_s\right)\right) \frac{\mathscr{D}_t \eta_t}{\eta_t} \\ &\quad - \left(1 - \omega\left(\eta_s\right)\right) \left(\frac{1}{\sigma_\delta} \int_t^s \mathscr{D}_t g_u \mathrm{d}\widehat{W}_u + \frac{1}{\sigma_\delta} \int_t^s g_u \mathscr{D}_t g_u \mathrm{d}u\right) \end{aligned}$$

which we substitute into (17). Simplifying yields (12). One still needs to compute the price S_t^T of a single-dividend paying stock in order to obtain (11) and (12). The latter is still given by the expression in Proposition 8 except that $\frac{S_t}{\delta_t}|_{M.C.}$ is replaced by the price of a single-dividend claim when Agent A is left alone in the economy, i.e. (13). The price-dividend ratio $\frac{S_t}{\delta_t}|_{O.U.}$ when Agent B is left alone in the economy is replaced by

$$e^{-\rho(T-t)+\alpha(T-t)+\beta_2(T-t)\left(\widehat{f}_t - g_t\right)}.$$

This expression is obtained as a particular case of Proposition 7 when the stock only pays off at a particular point T in time. Finally, the intermediate terms F^j now obey the following PDE

$$\widetilde{\mathscr{L}}^{\widehat{f},g} F^j + X^j F^j + \frac{\partial}{\partial t} F^j = 0$$

with boundary condition $F^j(T) = 1$. The latter PDE is derived as a special case of Proposition 6 when the stock only pays at a particular point T in time.

□

H Recursive Utility Formulation

In this section, we expose the formulation of the model when agents feature recursive utility preferences: Assume that Agents A and B have recursive preferences of the type:

$$V_t^i = E_t^i \left[\int_t^\infty f\left(c_s^i, V_s^i\right) \mathrm{d}s \right]$$

where the aggregator f is assumed to be of the Kreps and Porteus type:

$$f(c,v) = \frac{\beta}{\rho} \left[c^\rho (\gamma v)^{1-\frac{\rho}{\gamma}} - \gamma v \right].$$

We follow Dumas, Uppal, and Wang (2000) and reformulate the problem of Agent i in terms of a felicity function F as

$$\lambda_t^i V_t^i = \inf_{\nu^i} E_t^i \left[\int_t^\infty \lambda_s^i F\left(c_s^i, \nu_s^i\right) \mathrm{d}s \right]$$

subject to

$$\frac{\mathrm{d}\lambda_t^i}{\lambda_t^i} = -\nu_t^i \mathrm{d}t \quad \lambda_0^i = 1$$

where

$$F(c,\nu) = \frac{\beta}{\gamma} c^\gamma \left(\frac{\gamma - \frac{\rho}{\beta}\nu}{\gamma - \rho} \right)^{1-\frac{\rho}{\gamma}}.$$

As in the model exposed above, it is assumed that agents have homogenous preferences. It is then convenient to formulate the equilibrium in terms of the social planner problem. The social planner problem under the probability measure of agent A accordingly satisfies:

$$J \left(\sup_{c_1,c_2} \inf_{\nu_1,\nu_2} E \left[\int_0^\infty \left(\lambda_s^1 F\left(c_1, s, \nu_s^1\right) + \lambda_s^2 \eta_s F\left(c_2, s, \nu_s^2\right) \right) \mathrm{d}s \right] \right)$$

subject to

$$\frac{\mathrm{d}\lambda_t^i}{\lambda_t^i} = -\nu_t^i \mathrm{d}t \quad \lambda_0^i = 1$$

and the aggregate resource constraint

$$c_{1,t} + c_{2,t} = \delta_t$$

which is equivalent to

$$J\left(\sup_{c_1,c_2}\inf_{\nu_1,\nu_2} E\left[\int_0^\infty \left(\lambda_s^1 F\left(c_1,s,\nu_s^1\right)+\widetilde{\lambda}_s^2 F\left(c_2,s,\nu_s^2\right)\right)\mathrm{d}s\right]\right)$$

subject to

$$\begin{aligned}
\frac{\mathrm{d}\lambda_t^1}{\lambda_t^1} &= -\nu_t^1\mathrm{d}t \quad \lambda_0^1=1\\
\frac{\mathrm{d}\widetilde{\lambda}_t^2}{\lambda_t^2} &= -\nu_t^2\mathrm{d}t-\frac{g_t}{\sigma_\delta}\mathrm{d}W_t \quad \widetilde{\lambda}_0^2=1\\
c_{1,t}+c_{2,t} &= \delta_t.
\end{aligned}$$

The Hamilton-Jacobi-Bellman equation for this problem is given by

$$0=\sup_{c_1,c_2}\inf_{\nu^1,\nu^2}\lambda^1\left(F\left(c_1,\nu^1\right)-J_{\lambda^1}\nu^1\right)+\widetilde{\lambda}^2\left(F\left(c_2,\nu^2\right)-J_{\widetilde{\lambda}}\nu^1\right)+\mathcal{L}^{\delta,\lambda^1,\widetilde{\lambda}^2,\widetilde{f},g}.$$

The first order conditions are simply given by the convex conjugates:

$$\begin{aligned}
f\left(c_1,J_{\lambda^1}\right) &= F\left(c_1,\nu^1\right)-J_{\lambda^1}\nu^1\\
f\left(c_2,J_{\widetilde{\lambda}^2}\right) &= F\left(c_2,\nu^2\right)-J_{\lambda^2}\nu^2
\end{aligned}$$

and the optimal consumptions are given by

$$\begin{aligned}
c_1 &= \frac{\left(\gamma J_{\lambda^1}\right)^{\frac{1-\rho/\gamma}{1-\rho}}\left(\lambda^1\right)^{\frac{1}{1-\rho}}}{\left(\gamma J_{\lambda^1}\right)^{\frac{1-\rho/\gamma}{1-\rho}}\left(\lambda^1\right)^{\frac{1}{1-\rho}}+\left(\gamma J_{\widetilde{\lambda}^2}\right)^{\frac{1-\rho/\gamma}{1-\rho}}\left(\widetilde{\lambda}^2\right)^{\frac{1}{1-\rho}}}\delta\\
c_2 &= \frac{\left(\gamma J_{\widetilde{\lambda}^2}\right)^{\frac{1-\rho/\gamma}{1-\rho}}\left(\widetilde{\lambda}^2\right)^{\frac{1}{1-\rho}}}{\left(\gamma J_{\lambda^1}\right)^{\frac{1-\rho/\gamma}{1-\rho}}\left(\lambda^1\right)^{\frac{1}{1-\rho}}+\left(\gamma J_{\widetilde{\lambda}^2}\right)^{\frac{1-\rho/\gamma}{1-\rho}}\left(\widetilde{\lambda}^2\right)^{\frac{1}{1-\rho}}}\delta
\end{aligned}$$

Finally, we can use the homogeneity property arising from homogenous preferences and conjecture:

$$J\left(\delta,\lambda^1,\widetilde{\lambda}^2,\widetilde{f},g\right)=\frac{\delta^\gamma\left(\lambda^1+\widetilde{\lambda}^2\right)}{\gamma}j\left(\omega,\widetilde{f},g\right)$$

where

$$\omega=\frac{\lambda^1}{\lambda^1+\widetilde{\lambda}^2}.$$

Substituting this conjecture into the Hamilton-Jacobi-Bellman equation along with the first order conditions yields a simpler partial differential equation in 3 state variables. The boundary conditions in the dimensions $\widetilde{f}, g$ are absorbing as usual. The boundary conditions in the $\omega \in [0, 1]$ dimension are given by the value function when agent A and B are representative agents respectively. The price and the relevant equilibrium quantities are then obtained by decentralizing the equilibrium.

I Tables

	UD	DD	DDU	DDD	DU
$\widehat{f}$	-0.03	-0.08	-0.095	-0.095	-0.03
$\widehat{f}^B$	$\bar{f}$	$\bar{f}$	-0.03	-0.04	-0.03

Table 1: State Variables Values Characterizing the Five Phases of a Bearish Cycle.
Each phase of the bearish cycle is characterized by a pair of values for the filter $\widehat{f}$ of Agent A and the filter $\widehat{f}^B$ of Agent B or, equivalently, the disagreement g. The phase UD is characterized by a first streak of bad news in the observable shock $\widehat{W}$. The phase DD represents a slight deterioration of the situation with respect to UD: it captures the difference in the adjust speed of the beliefs of both agents. Phases DDU and DDD respectively are interpreted as periods of a deep move into the bearish cycle and the worst phase of this episode. Finally, case DU denotes the exit of the bearish cycle, a point at which both agents's expectations meet again.

$\bar{f}$	κ	σ_f	f^s	f^d	ψ	λ	σ_δ	ρ	α
0.02	0.2	0.015	0.03	-0.1	0.5	0.04	0.03	0.01	3

Table 2: Calibration.
$\bar{f}$ is the long-term mean, κ the mean-reversion speed, and σ_f the volatility of the Ornstein-Uhlenbeck process. f^s and f^d are the up and down states of the Markov chain, respectively. ψ and λ are the downward and upward probabilities of the Markov chain, respectively. σ_δ is the dividend volatility, ρ the subjective discount rate, and α the relative risk aversion.

	Estimate	t-Statistic	P-Value	R^2
α	-0.000162	-0.121737	0.903181	0.1001
β_1	-0.006144	-0.646775	0.518222	
β_2	-0.071288	-3.315547	0.001015	

Table 3: Impulse Response Regression Outputs (i)**.**
We regress the contemporaneous S&P500 returns on the changes in the Google search volume index. Parameter estimates, t-statistics, and P-values corresponding to case (i)'s regression. Good and bad times are defined using GDP growth forecasts.

	Estimate	t-Statistic	P-Value	R^2
α	-0.000406	-0.291461	0.770879	0.0871
β_1	-0.014682	-2.285479	0.022911	
β_2	-0.054044	-2.381808	0.017787	

Table 4: Impulse Response Regression Outputs (*ii*).
We regress the contemporaneous S&P500 returns on the changes in the Google search volume index. Parameter estimates, t-stats, and P-values corresponding to case (*ii*)'s regression. Good and bad times are defined using GDP growths.

	Estimate	t-Statistic	P-Value	R^2
α	0.085	239.7940	0	0.0850
β_1	-0.0212	-11.4605	0	

Table 5: Linear Fit of the Risk Reversal against the Risk Free Rate.
Parameter estimates, t-statistics, and P-values corresponding to the linear fit between the model implied risk reversal and the model implied risk free rate.

	Estimate	t-Statistic	P-Value	R^2
α	0.0004	12.8795	0	0.0403
β_1	-0.0020	-14.063	0	

Table 6: Linear Fit of the Risk Reversal against the Stock Return.
Parameter estimates, t-statistics, and P-values corresponding to the linear fit between the model implied risk reversal and the model implied stock return. The results indicate a strong negative relation between these two equilibrium quantities.

J Figures

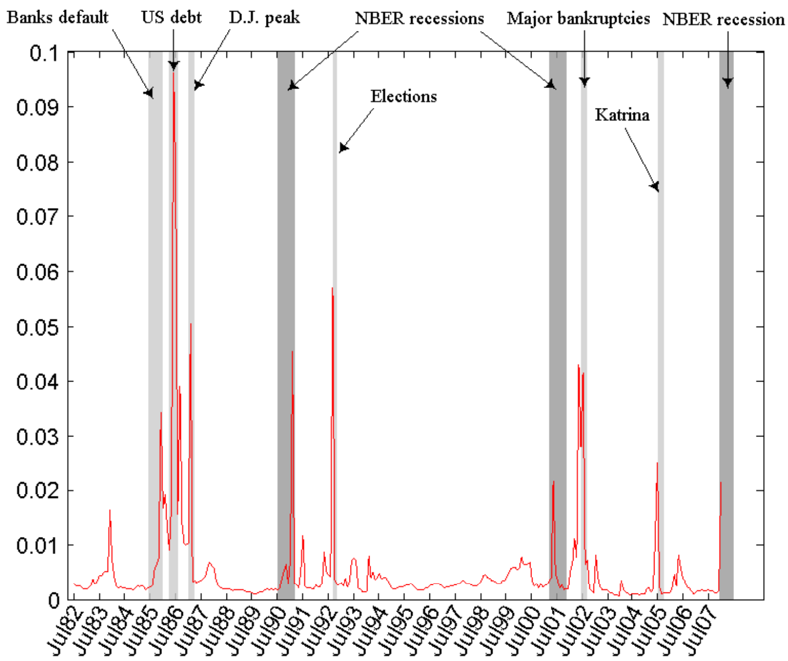
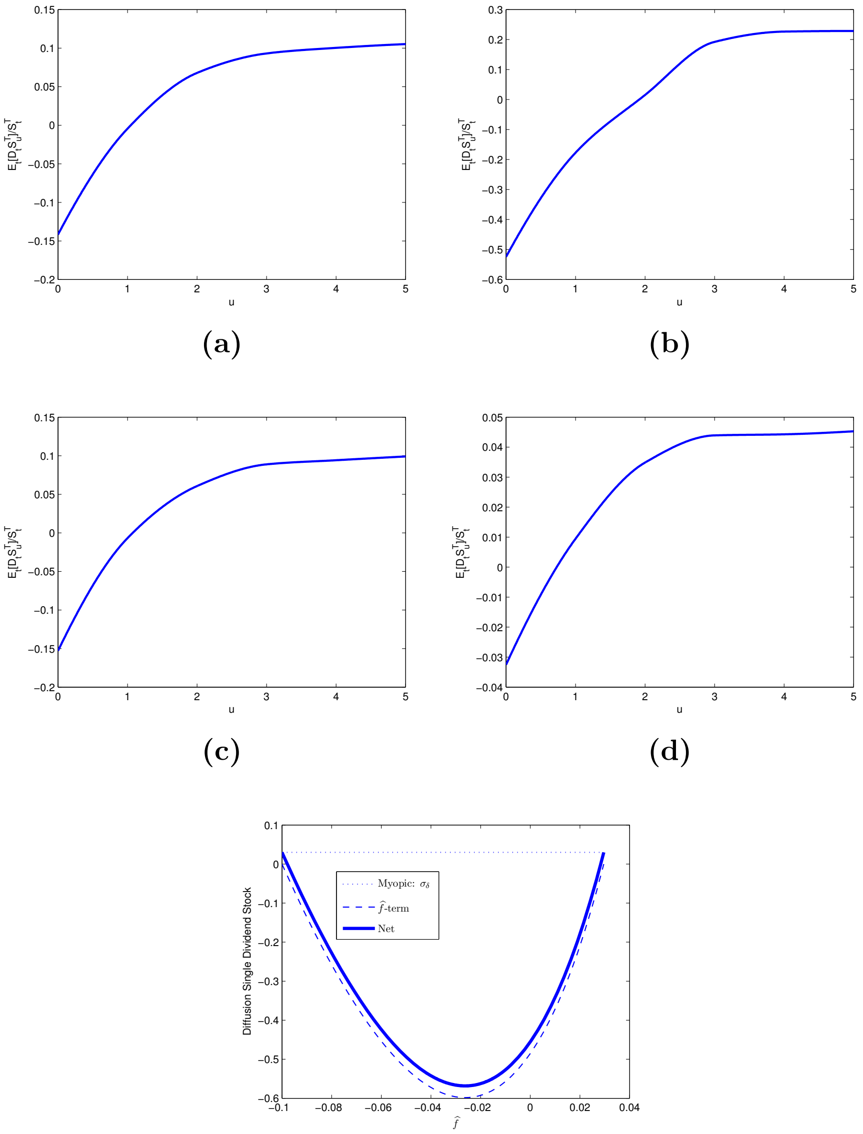


Figure 1: Analyst Forecasts Dispersion.
Monthly data from 1982 to 2007 are provided by IBES and CRSP. The dispersion index is constructed as follows: We record US firms that are followed by more than 5 analysts on each month of the period. This makes a total of 29 firms. Let's call f^i_{jt} the forecast of analyst i at time t regarding firm j's earnings per share, $\mu_{jt} = \frac{1}{N_t}\sum_{i=1}^{N_t} f^i_{jt}$ the time t average forecast for firm j, $\sigma^2_{jt} = \frac{1}{N_t}\sum_{i=1}^{N_t}\left(f^i_{jt} - \mu_{jt}\right)^2$ the time t variance of firm j's forecast, and p_{jt} the price of stock j at time t. The measure of the analysts forecast's dispersion d_t is defined by $d_t = \frac{1}{29}\sum_{j=1}^{29}\frac{\sigma_{jt}}{p_{jt}}$.
Events are principally collected from Campbell Harvey's and NBER's websites. Year 1985: "A record 43,000 farms go bankrupt as land prices fall and interest rates soar. Many banks and savings-and-loan institutions go bankrupt in Texas, Oklahoma, and other oil states that are pressured by collapsing world oil prices". Summer 1986: US debt hits 2000 billion dollars. January 1987: "The stock market continues to rise, with the Dow Jones closing at 2002.25". July 1990-March 1991: NBER recession. November 1992: "Bill Clinton is elected President". March 2001-November 2001: NBER recession. Summer 2002: WorldCom and US Airways go bankrupt. Summer 2005: The US are hit by several hurricanes and tornados. Katrina's estimated damage is 81 billion dollars. December 2007-June 2009: NBER recession. We are very grateful to Tony Berrada for providing clean, extensive, and meaningful analyst forecasts data.



(a) **(b)** **(c)** **(d)** **(e)**

Figure 2: Price Impulse Response in the Benchmark Model. Expectation of the future relative price sensitivity to a current shock in the Brownian motion. Figure 2a corresponds to $\widehat{f} = \bar{f}$, 2b to $\widehat{f} = -0.03$, 2c to $\widehat{f} = -0.08$, and 2d to $\widehat{f} = -0.095$. Figure 2e is the volatility of the single-dividend paying stock against $\widehat{f}$. Dotted is "myopic", dashed is future $\hat{f}$ "hedging", and solid is net.

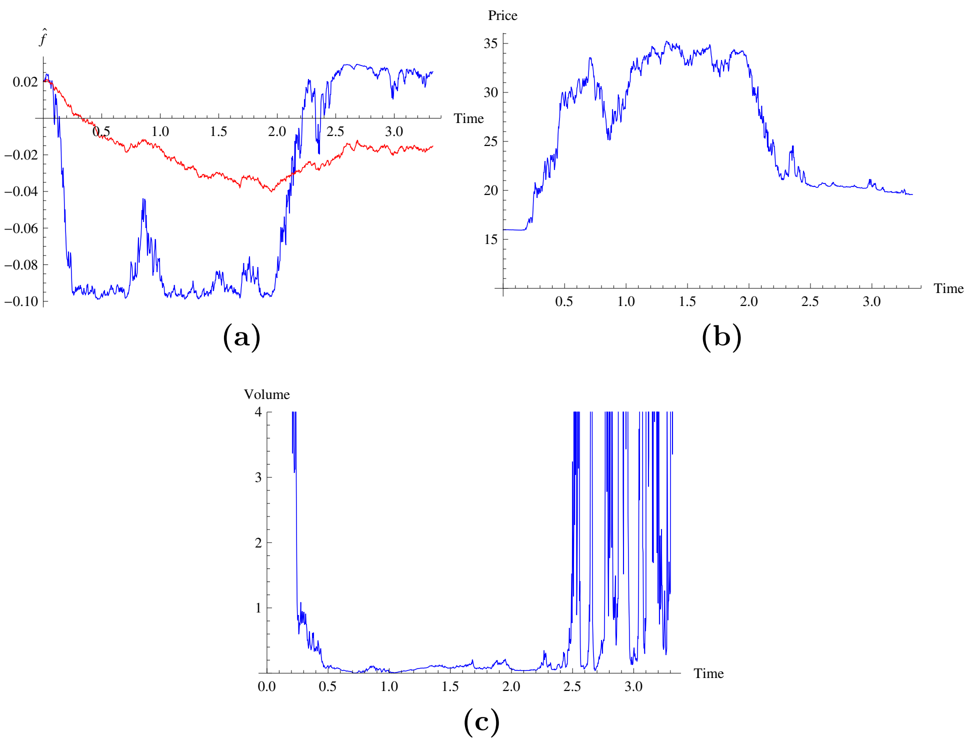


Figure 3: Simulation of a Perceived Crash.
Figure 3a exhibits a simulated path of $\widehat{f}$ in blue and $\widehat{f}^B$ in red. Figure 3b plots the associated simulated price path. Figure 3c plots the associated simulated volume.

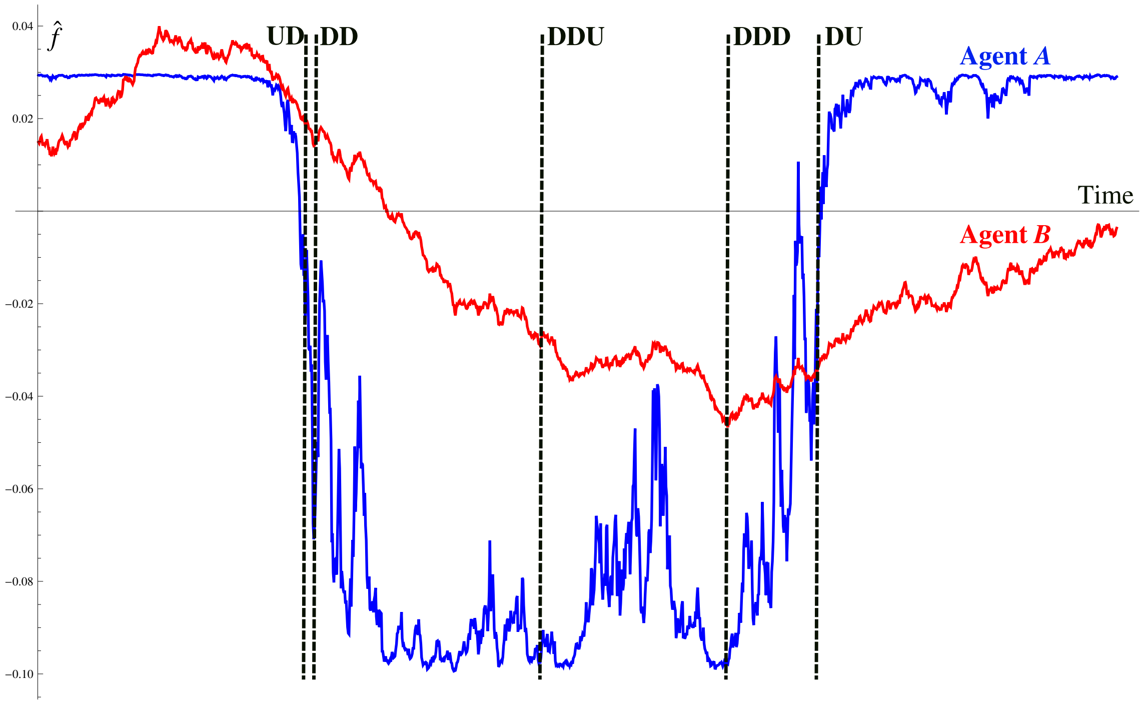


Figure 4: The Five Different Phases of a Bearish Cycle.
Filtered fundamental perceived by Agent A and Agent B, respectively.

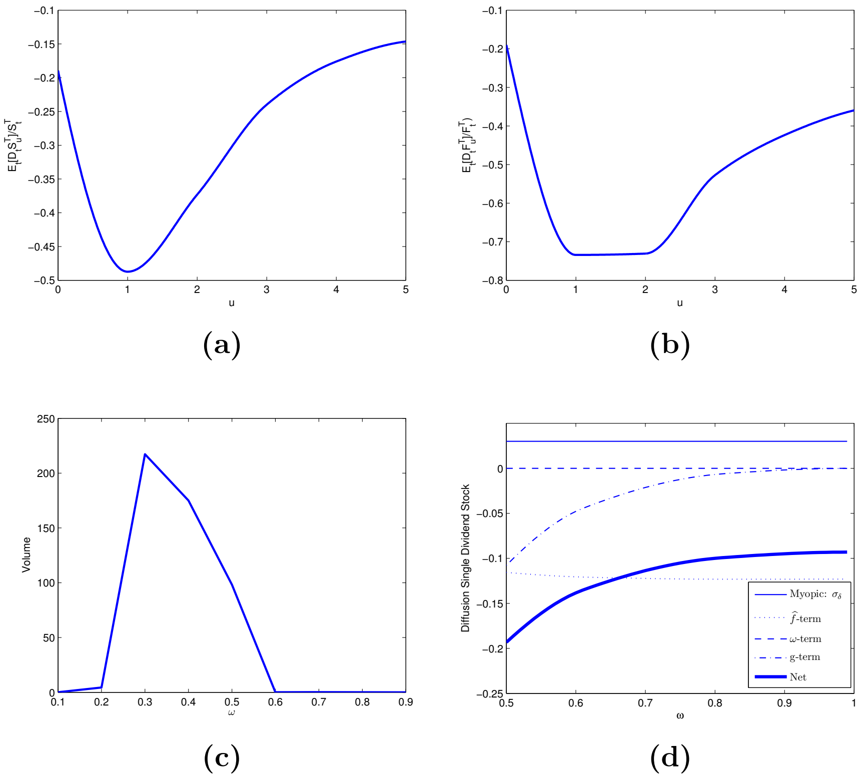


Figure 5: Price and Wealth Impulse Responses, Trading Volume, and Volatility in the Status Quo.
The top figures represent the expectation of the future relative price sensitivity and future relative wealth sensitivity to a current shock in the Brownian motion. The bottom figures are the volume and the volatility of the single-dividend paying stock against ω, respectively. Thin solid is "myopic", dotted is future $\hat{f}$ "hedging", dashed is future ω "hedging", dash dotted is future g "hedging", and thick solid is net. The calibration is: $(\widehat{f}, \widehat{f}^B) = (\bar{f}, \bar{f})$ and $\omega = 0.5$.

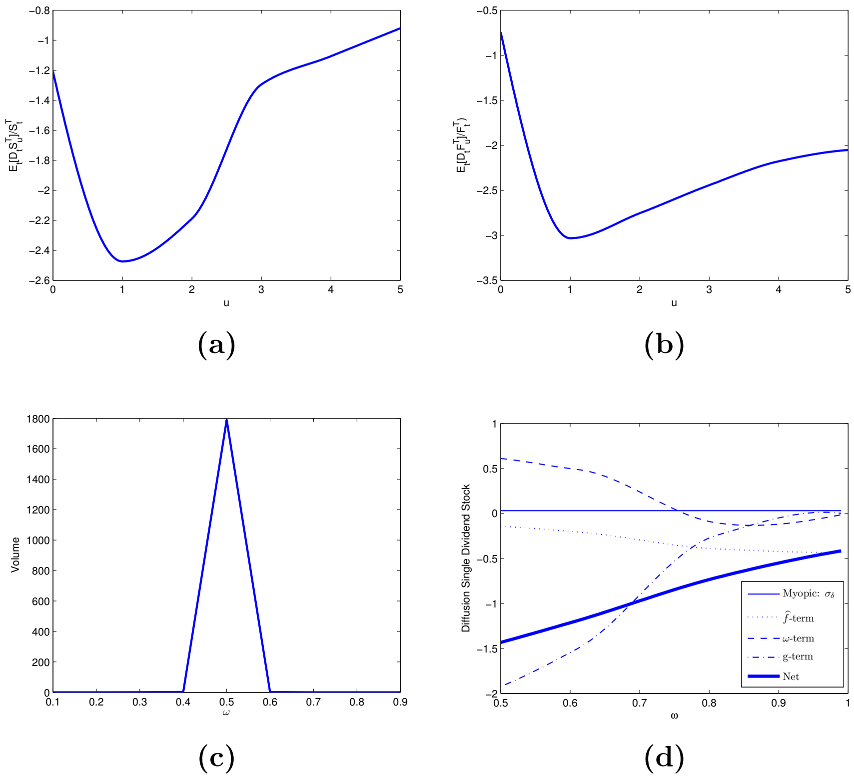


Figure 6: Price and Wealth Impulse Responses, Trading Volume, and Volatility in Phase UD.

The top figures represent the expectation of the future relative price sensitivity and future relative wealth sensitivity to a current shock in the Brownian motion. The bottom figures are the volume and the volatility decomposition of the single-dividend paying stock against ω, respectively. Thin solid is "myopic", dotted is future $\hat{f}$ "hedging", dashed is future ω "hedging", dash dotted is future g "hedging", and thick solid is net. The calibration is: $(\widehat{f}, \widehat{f}^B) = (-0.03, \bar{f})$ and $\omega = 0.5$.

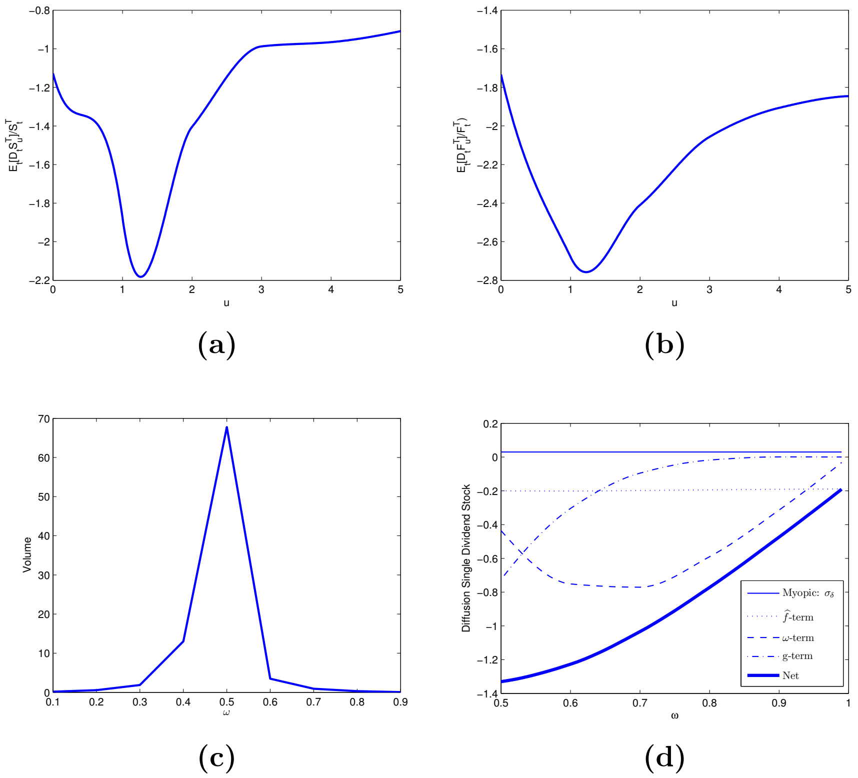


Figure 7: Price and Wealth Impulse Responses, Trading Volume, and Volatility in Phase DD.
The top figures represent the expectation of the future relative price sensitivity and future relative wealth sensitivity to a current shock in the Brownian motion. The bottom figures are the volume and the volatility decomposition of the single-dividend paying stock against ω, respectively. Thin solid is "myopic", dotted is future $\hat{f}$ "hedging", dashed is future ω "hedging", dash dotted is future g "hedging", and thick solid is net. The calibration is: $(\widehat{f}, \widehat{f}^B) = (-0.08, \bar{f})$ and $\omega = 0.5$.

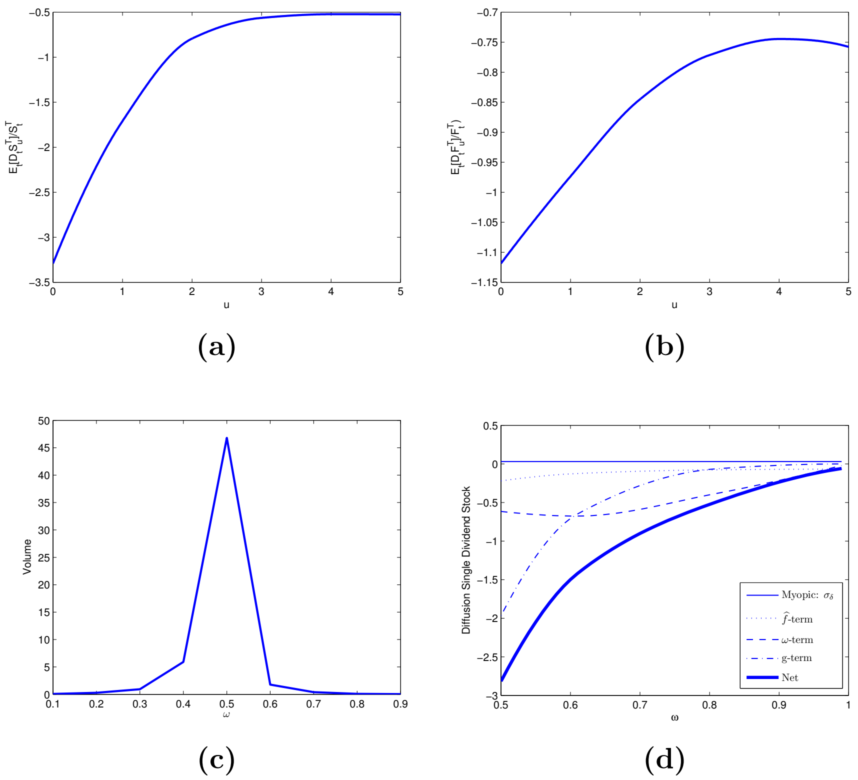


Figure 8: Price and Wealth Impulse Responses, Trading Volume, and Volatility in Phase DDU.

The top figures represent the expectation of the future relative price sensitivity and future relative wealth sensitivity to a current shock in the Brownian motion. The bottom figures are the volume and the volatility decomposition of the single-dividend paying stock against ω, respectively. Thin solid is "myopic", dotted is future $\hat{f}$ "hedging", dashed is future ω "hedging", dash dotted is future g "hedging", and thick solid is net. The calibration is: $(\widehat{f}, \widehat{f}^B) = (-0.095, -0.03)$ and $\omega = 0.5$.

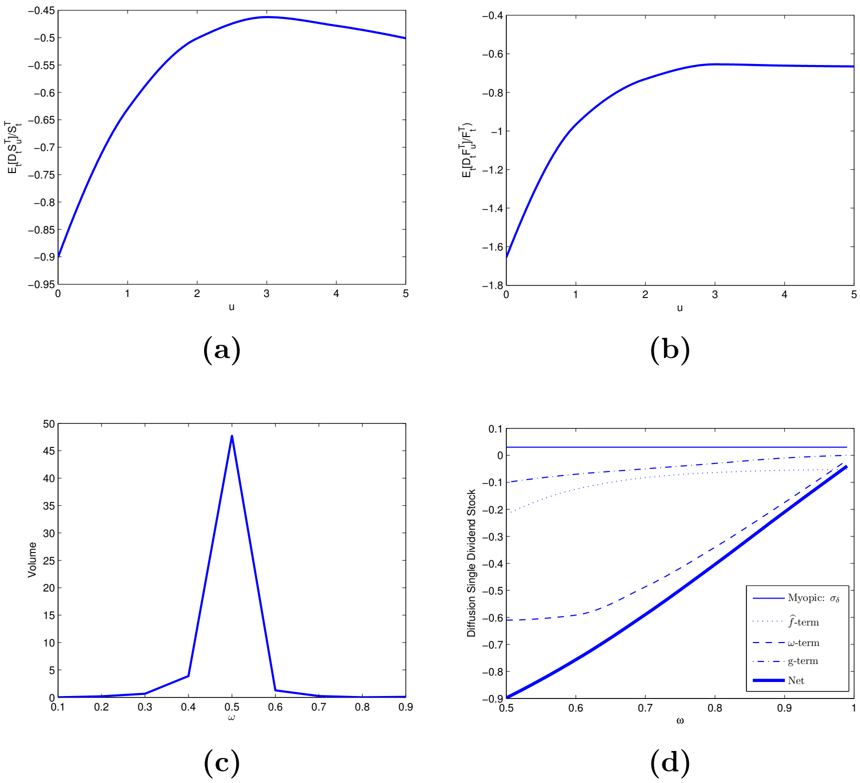


Figure 9: Price and Wealth Impulse Responses, Trading Volume, and Volatility in Phase DDD.

The top figures represent the expectation of the future relative price sensitivity and future relative wealth sensitivity to a current shock in the Brownian motion. The bottom figures are the volume and the volatility decomposition of the single-dividend paying stock against ω, respectively. Thin solid is "myopic", dotted is future $\hat{f}$ "hedging", dashed is future ω "hedging", dash dotted is future g "hedging", and thick solid is net. The calibration is: $(\widehat{f}, \widehat{f}^B) = (-0.095, -0.04)$ and $\omega = 0.5$.

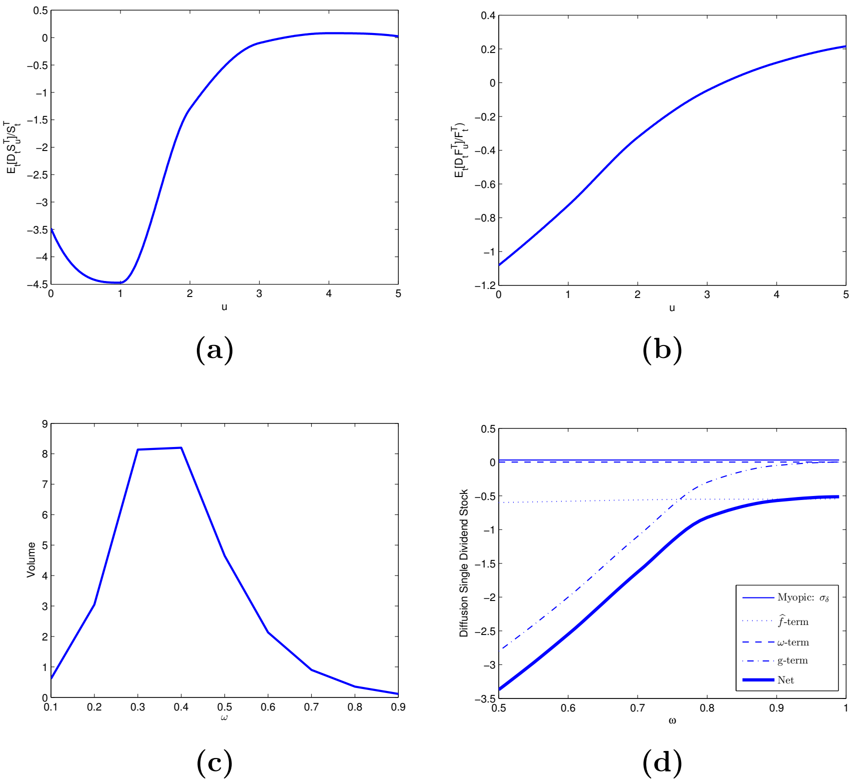


Figure 10: Price and Wealth Impulse Responses, Trading Volume, and Volatility in Phase DU.
The top figures represent the expectation of the future relative price sensitivity and future relative wealth sensitivity to a current shock in the Brownian motion. The bottom figures are the volume and the volatility decomposition of the single-dividend paying stock against ω, respectively. Thin solid is "myopic", dotted is future $\hat{f}$ "hedging", dashed is future ω "hedging", dash dotted is future g "hedging", and thick solid is net. The calibration is: $(\widehat{f}, \widehat{f}^B) = (-0.03, -0.03)$ and $\omega = 0.5$.

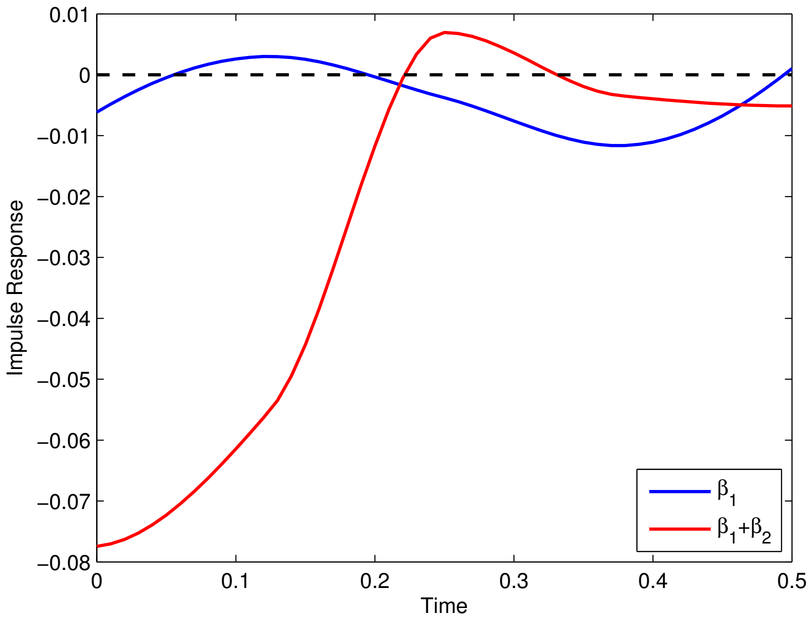


Figure 11: S&P500 Impulse Response (i)**.**
Response of the S&P500 to a change in the search volume index between 2004 to 2010. The value weighted search volume index is built using Google search volumes on: financial crisis, crisis, crises, recession, and bankruptcy. Good times (bad times) are defined by the periods in which the GDP growth forecasts are above (below) their average.

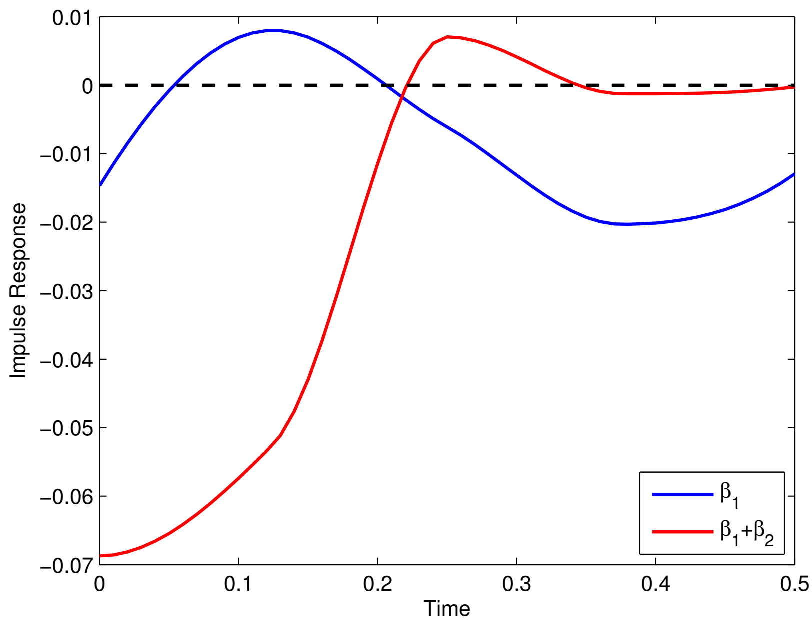


Figure 12: S&P500 Impulse Response (*ii*)**.**
Response of the S&P500 to a change in the search volume index between 2004 to 2010. The value weighted search volume index is built using Google search volumes on: financial crisis, crisis, crises, recession, and bankruptcy. Good times (bad times) are defined by the periods in which the GDP growths are above (below) their average.

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