The Low-Minus-High Portfolio and the Factor Zoo

Daniel Andrei* Julien Cujean† Mathieu Fournier‡

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Abstract

Anomalies in the cross section of returns should not be regarded as evidence against the CAPM. Regardless whether the CAPM is rejected for valid reasons or by mistake, a single long-short portfolio will always explain, together with the market, 100% of the cross-sectional variation in returns. Yet, this portfolio need not proxy for fundamental risk. We show theoretically how factors based on valuation ratios (e.g, book-to-market), or on investment rates, can be proxies for this portfolio. More generally, the empiricist can uncover an infinity of proxies for this portfolio, thus unleashing the factor zoo.

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*McGill University, Desautels Faculty of Management, Office 549, 1001 Sherbrooke Street West, Montréal, Québec, H3A 1G5, Canada, danielandrei.info, daniel.andrei@mcgill.ca.

†University of Berne, Institute for Financial Management, Engehaldenstrasse 4, 3012 Bern, Switzerland, juliencujean.com, julien.cujean@ifm.unibe.ch.

‡HEC Montréal, Office 4230, 3000 Chemin de la Côte-Sainte-Catherine, Montréal, Québec, H3T 2B1, Canada, mathieufournier.net, mathieu.fournier@hec.ca.

"We really don't know whether to believe the theory or the data... In the light of the uncertainty about the reasons for the difference between the theory and the data, the safest course may be to assume that the theory is correct."

— *Black and Scholes (1974, p. 405)*

1 Introduction

The empirical analysis of the cross-section of returns is a significant achievement of half a century of finance research. Perhaps the most tested hypothesis in this area of research is the question of whether beta as a sole variable can explain returns—the main prediction of the Capital Asset Pricing Model. Early tests of this prediction led to its partial empirical validation,[1] then to its repeated demise.[2] Not only returns and betas are not related as the CAPM predicts, but empiricists have also uncovered a "zoo" of additional factors. After half-century of research, empirical asset pricing has reached an uncomfortable impasse, examining hundreds of anomalies and arguing over their interpretations.[3]

This state of affairs, where finance researchers postulate more and more factors to explain the cross-section of returns, reminds us of the situation reached by early astronomers, who kept adding epicycles to explain the movements of celestial bodies. Is empirical asset pricing down the path of "adding epicycles"?

In this paper, we argue that rejecting the CAPM on the basis that other factors besides beta explain the cross-section of returns is not a valid test of the theory. When the CAPM is rejected—for valid reason(s) or by mistake—there will *always* be a long-short portfolio (i.e., a factor) that explains, together with the market, the cross-section of returns. This portfolio will become empiricist's strongest ally against the CAPM: even when this portfolio is devoid of economic meaning, the empiricist will fail to reject a multifactor model of returns. To add to the confusion, as soon as any observable variable (macroeconomic factor, or firm characteristic) covaries with this portfolio, this variable becomes a contender to join the zoo. One can thus find an infinity of anomalies, forever adding epicycles.

Our argument is mainly theoretical. Nevertheless, before delving into theory, it might be helpful to begin with an example. We develop our example in a set of portfolios that has become the "playing field" of empirical asset pricing: the 25 size and book/market sorted portfolios (Fama and French, 1993). The spectacular failure of the CAPM in this set of portfolios is now a textbook example (e.g., Cochrane, 2009; Campbell, 2017); one can

[1] Blume and Friend (1973), Fama and MacBeth (1973).

[2] Reinganum (1981), Lakonishok and Shapiro (1986), Fama and French (1992, 1993). See Fama and French (2004) for a comprehensive review.

[3] Harvey, Liu, and Zhu (2016), Hou, Xue, and Zhang (2018).

perhaps say that this 5×5 portfolio space is the coffin of the CAPM.

In this portfolio space, we build a long-short portfolio that, together with the market, explains 89% of the cross-sectional variation of returns (in comparison, the Fama and French (1993) three-factor model explains 63% of the variation, whereas the Fama and French (2015) five-factor model explains 73%). Theoretically, the long-short portfolio represents the difference between the full-sample mean-variance efficient portfolio with the highest Sharpe ratio (in mean-variance terms, the *tangency* portfolio) and the market portfolio. For an econometrician who observes data ex-post, this long-short portfolio represents a way to improve efficiency of the market portfolio: assets with positive weights in this portfolio are *under*-invested (cheap) and assets with negative weights are *over*-invested (expensive). Thus, we denote the difference between the tangency and the market portfolios as the *Low-Minus-High* (LMH) portfolio.

Of course, in *any* sample of returns, one can *always* find a tangency portfolio that generates exact linearity between betas and expected returns (Roll, 1977); the LMH portfolio, which is built based on the tangency portfolio, will tautologically benefit from this perfect hindsight. In our example, however, we build the LMH portfolio in the same way as any other risk factor is commonly built, with data that are available only at the time of portfolio formation. Yet, we show that none of the five risk factors from Fama and French (2015) or the momentum factor from Carhart (1997) can explain the returns of the LMH portfolio: its alpha is above 1% per month, independently of the factors used as control variables. Instead, the alphas of existing factors mostly disappear when we regress their returns on the returns of the LMH portfolio. Furthermore, the LMH portfolio commands a positive and strongly statistically significant risk premium, alone or when controlling for the market and/or any other factor(s). Finally, the LMH portfolio has significant explanatory power in other portfolio sorts (e.g., sorts based on past returns).

Have we found, yet again, a better factor, a mighty inhabitant of the zoo? What are the macroeconomic risks (if any) underlying the LMH portfolio? To address these questions, we appeal to the authority of theory. Our starting point is an equilibrium model in which the CAPM holds—in the words of Fischer Black and Myron Scholes, we will "assume that the theory is correct" (Black and Scholes, 1974, p. 405). In this equilibrium model, investors trade based on private and public information and, on aggregate, hold the market portfolio. The econometrician, who does not observe the information of investors, mis-measures betas and rejects the CAPM (Andrei, Cujean, and Wilson, 2018). Thus, we start from the premise that the CAPM is rejected by mistake.

We then let the econometrician build the LMH portfolio and show that indeed this portfolio explains, together with the market, 100% of the cross-section of returns. The econometrician, thus, fails to reject a two-factor model of returns. Moreover, we show that observable characteristics, such as market-to-book ratios or investment rates, are perfect proxies for the LMH portfolio. In the eyes of the econometrician, firms with low market-to-book ratios or firms with low investment rates command a positive risk premium, in addition to the premium earned from exposure to the market alone. Yet, in the model there is no economic reason for firms with low market-to-book ratio or low investment rates to appear relatively riskier than the CAPM predicts. More generally, the econometrician can uncover an infinity of proxies for the LMH portfolio, thus unleashing the factor zoo.

We conclude that anomalies in the cross section of returns should not be regarded as evidence against the CAPM. The fact that one single factor can account for the (possibly spurious) mispricing of the CAPM lures empiricists into a methodological trap. To be sure, we, as many others, believe that many factors are very important and puzzling; and nothing in our paper undermines their high risk-adjusted returns. However, our objective is to judge whether this approach of looking for factors makes a good case against the CAPM. Because this approach will always lead to discovery of factors—economically meaningful or not—it cannot be regarded as a test of the CAPM.

Roll (1977) has argued that the CAPM will perhaps never be tested. Berk (1995) has argued that the size anomaly cannot be regarded as evidence against any asset pricing theory. We argue that anomalies *in general* should not be regarded as evidence against the CAPM, independently of whether they earn high risk-adjusted returns or not.

The paper proceeds as follows. In the next section we describe the LMH portfolio in the 25 *Size-B/M* portfolio space. Section 3 develops our theoretical model. Section 4 presents further empirical results, and Section 5 concludes.

2 The Low-Minus-High Portfolio

Figure 1 presents mean excess returns and volatilities for 25 size and book/market sorted portfolios (Fama and French, 1993), in monthly data from 7/1963 to 12/2018, 666 observations. Panel (a) plots these portfolios in a mean-standard deviation diagram, together with the full-sample efficient frontier. The triangle labeled M is the market portfolio, whereas the dot labeled T is the tangency portfolio, computed based on the entire sample.

Panel (b) illustrates the empirical failure of the CAPM. It plots the expected excess returns of the above 25 portfolios against market betas. Return and beta are not related as the CAPM dictates—the Security Market Line (SML) has a negative slope and a statistically significant positive intercept (numbers are provided in the caption of the figure).

The rejection of the CAPM can also be interpreted in terms of the geometric distance

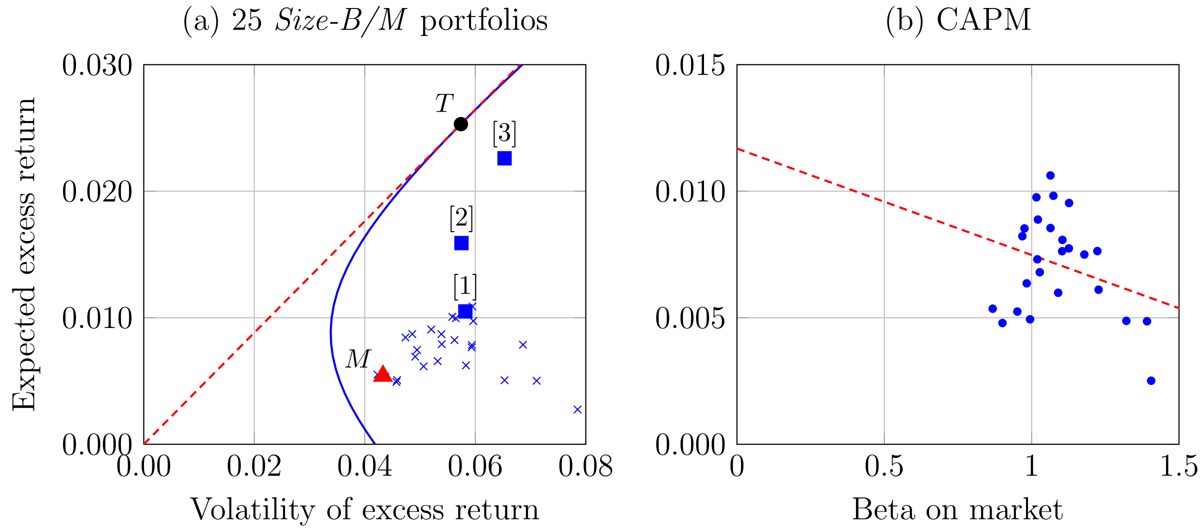

Figure 1: **25 *Size-B/M* portfolios and the CAPM**
Panel (a): Monthly mean excess returns and volatilities for 25 *Size-B/M* sorted portfolios (Fama and French, 1993), July 1963-December 2018, 666 months. The solid line is the (ex-post) minimum-variance frontier and the dashed line is the CML. The squares labeled [1], [2], and [3] are defined in the text. Panel (b): Mean excess returns against beta on market (the market portfolio is depicted in panel (a) with the red triangle labeled M). The dashed line is the SML, which has an intercept of 0.0117 (t-stat 3.01), a slope of -0.0042 (t-stat -1.00), and a coefficient R^2 of 4.8%.

between the portfolios M and T. It is obvious from panel (a) that this distance is large: the expected excess return of portfolio T is orders of magnitude larger than the expected excess return of the market. Thus, a test based on this distance (Gibbons, Ross, and Shanken, 1989) will likely reject the CAPM, and panel (b) confirms.

The rejection of the CAPM illustrated above has led to a decades-long quest for alternative factors. This can be best understood in the familiar mean-variance diagram: when factors are long-short portfolios with zero net investment—e.g., “Small-Minus-Big” (SMB), “High-Minus-Low” (HML)—one can simply add these portfolios to the market and obtain a *tilted* portfolio, with the hope that this latter portfolio does a better job for explaining asset returns. Consider, for instance, the square labeled [1] in panel (a) of Figure 1. It represents the tilted portfolio after adding the Fama and French (1993) SMB and HML factors. The square labeled [2] adds two more factors, “Robust-Minus-Weak” (RMW) and “Conservative-Minus-Aggressive” (CMA) (Fama and French, 2015). Finally, the square labeled [3] further adds the momentum factor (Carhart, 1997). One can therefore think of the search for factors as an effort to improve the Sharpe ratio: as more factors are added, there is a clear tendency of the tilted portfolio to move upward towards the portfolio with the maximum Sharpe ratio attainable theoretically, T.

We will adopt a more pragmatic approach. Instead of gradually trying to move towards the tangency portfolio T, we start directly from T. More precisely, let M and T be the vectors of portfolio weights for the market and the tangency portfolios, and consider the portfolio of deviations between T and M (the portfolio T is likely composed of short positions, but this does not change our argument):

$$\Delta = T - M. \tag{1}$$

Since both T and M sum up to one, Δ sums up to zero and is therefore a long-short portfolio. For an econometrician who observes data ex-post, Δ represents a way to improve efficiency of the market portfolio (with perfect hindsight). Assets with positive weights in this portfolio (Δ^+ assets) are *under*-invested; assets with negative weights in this portfolio (Δ^- assets) are *over*-invested. For the econometrician, under-invested assets appear relatively cheaper than over-invested assets. Thus, the portfolio Δ is a *Low-Minus-High* portfolio.

The econometrician can use the market portfolio M and the Δ portfolio to explain ex-post the entire cross-sectional variation in expected excess returns.

Proposition 1. *Consider an economy with N assets in which an econometrician observes time series of realized excess returns for all assets and for the market. Let the covariance matrix of excess returns for the N assets be Σ and their expected excess returns be μ. Then,*

$$\mu = \frac{\mu_T \sigma_M^2}{\sigma_T^2}\beta + \frac{\mu_T \sigma_\Delta^2}{\sigma_T^2}\beta_\Delta, \tag{2}$$

where $\mu_T \equiv \mu' T$, $\sigma_M^2 \equiv M'\Sigma M$, $\sigma_T^2 \equiv T'\Sigma T$, $\sigma_\Delta^2 \equiv \Delta'\Sigma\Delta$, $\beta \equiv \Sigma M/\sigma_M^2$, and $\beta_\Delta \equiv \Sigma\Delta/\sigma_\Delta^2$.

Proof. Since portfolio T is ex-post efficient, efficient set mathematics imply

$$\mu = \frac{\mu_T}{\sigma_T^2}\Sigma T. \tag{3}$$

Replacing $T = M + \Delta$ and using the definitions of β and β_Δ yields (2). □

Two sets of betas, one computed on the market portfolio M and one computed on the Δ portfolio, explain ex-post 100% of the variation in expected excess returns. A similar point has been previously made by Roll (1977, p. 138, emphasis his): "there will always be *some* portfolio which is ex-post efficient and will bring about exact observed linearity among ex-post sample mean returns and ex-post sample betas." In our context, this portfolio is T. Notice, however, that we are not interested in computing betas on T. Instead, we compute two sets of betas (β and β_Δ), because this allows us to understand how exactly the CAPM is rejected when considering the market and the LMH portfolios together.

According to Eq. (2), assets earn a positive risk premium $\mu_T\sigma_M^2/\sigma_T^2$ per unit of β on the market and a positive risk premium $\mu_T\sigma_\Delta^2/\sigma_T^2$ per unit of β_Δ on the Δ portfolio.[4] The question is, then, which assets have a high β_Δ. Because the Δ portfolio is a long-short portfolio, a fortuitous simplification yields a positive relation between β_Δ and Δ:

$$\beta_\Delta = \bar{\beta}_\Delta + \frac{1}{\Delta'\Delta}\Delta + u, \tag{4}$$

where $\bar{\beta}_\Delta$ is the arithmetic average over β_Δ and u is uncorrelated with Δ.[5]

We thus reach the following result. Low-demand assets (Δ^+ assets) tend to have a high β_Δ, whereas high demand assets (Δ^- assets) tend to have a low β_Δ. Together with Eq. (2), Eq. (4) implies that under-valued assets (in econometrician's view) earn a positive risk premium relatively to over-valued assets. This risk premium is not explained by the market factor. The econometrician, thus, uncovers a low-minus-high portfolio that has additional explanatory power for the cross-section of returns.

To be sure, the Δ portfolio appears to explain returns only if the CAPM does *not* hold or, in other words, if the market portfolio is not mean-variance efficient. It is easily verified that when $M = T$ the CAPM relation holds and the market factor explains the entire cross-sectional variation in expected excess returns. But when the CAPM is rejected, Proposition 1 implies that the rejection arises in a specific way: there will always be a low-minus-high portfolio that earns an unambiguously positive return.

Furthermore, the relation (2) is tautological in the sense that it is uninformative about the validity of the CAPM. The only assumptions necessary for Proposition 1 are that the covariance matrix Σ is non-singular and that at least one asset has a different sample mean return from others—which, incidentally, are assumptions (A.1) and (A.2) in Roll (1977). Aside from these two assumptions, we do not need to impose assumptions about investor preferences or to characterize the dynamics of assets' excess returns.

How close to the portfolio Δ can an econometrician get without perfect hindsight? To answer this question, we build a time-varying low-minus-high portfolio not with the full sample of returns, but with data that are available only at the time of portfolio formation.

[4]The two sets of betas in Eq. (1) results from two univariate first-pass regressions, as opposed to one multivariate regression of asset returns on the market and the Δ portfolio. A multivariate regression will yield linear combinations of β and β_Δ, with different slopes, but same R^2 (see Corollary 1.1).

[5]Eq. (4) results from a least squares calculation. Consider the $N \times 2$ matrix $X = [\mathbf{1}_N \, \Delta]$ and compute the estimated coefficients as $(X'X)^{-1}X'\beta_\Delta$. Since Δ is a long-short portfolio, we have $\mathbf{1}_N' \Delta = 0$ which simplifies the algebra and yields (4). Notice also that the arithmetic average $\bar{\beta}_\Delta$ is different from 1 (instead, the weighted average $\Delta'\beta_\Delta$ equals one).

	LMH	MKT	HML	SMB	RMW	CMA	MOM
LMH		-0.40	0.45	-0.06	0.06	0.38	0.11
MKT	-0.40		-0.24	0.26	-0.19	-0.32	-0.09
HML	0.46	-0.24		-0.14	-0.20	0.68	-0.16
SMB	-0.02	0.27	-0.19		-0.27	-0.16	0.01
RMW	0.08	-0.21	0.06	-0.39		-0.20	0.17
CMA	0.43	-0.37	0.70	-0.17	-0.04		-0.08
MOM	0.17	-0.14	-0.19	0.00	0.11	-0.03	

Table 1: **Correlations**
This table presents the time-series correlations between the returns of the LMH portfolio and returns of the MKT, HML, SMB, RMW, CMA, and MOM factors, July 1963-December 2018, 666 months. The below-diagonal entries show Pearson product-moment correlations. The above-diagonal entries show Spearman rank correlations.

That is, we build this portfolio in the same way as other risk factors are commonly built. We do not claim that in this way we build an *ex-ante* low-minus-high portfolio; rather, we say that this portfolio is as much "ex-ante" as any other risk factor is. We recognize that the mean-variance frontier depicted in Figure 1 may have little to do with the true, or ex-ante mean-variance frontier, and there is no reason to believe that this portfolio will preserve the properties of the ex-post portfolio Δ.

The procedure is as follows. At every time t starting from 6/1963, we use the past 30 years of monthly excess returns up to and including time t, to compute mean excess returns, μ_t, and the covariance matrix of excess returns, Σ_t. Relying on efficient set mathematics, we compute the tangency portfolio $T_t = \Sigma_t^{-1}\mu_t/B_t$, where $B_t \equiv \mathbf{1}'_N\, \Sigma_t^{-1}\mu_t$ and $N = 25$. Then we compute Δ_t as the difference between T_t and the mean market capitalizations for the 25 portfolios over the past 30 years of monthly data. Using Δ_t, we compute the one-month ahead excess return of the low-minus-high portfolio (from t to $t+1$). This yields a time series of monthly excess returns from 7/1963 to 12/2018. In the remaining of the paper, we refer to this time-varying portfolio as the *Low-Minus-High (LMH) portfolio*.

During the period from 7/1963 to 12/2018, the LMH portfolio produces a mean monthly excess return of 1.49% and a monthly standard deviation of 7.6%. The annualized Sharpe ratio for the monthly returns of the LMH portfolio is therefore 0.68. The correlations of the LMH portfolio returns with the Fama and French (2015) five factors (MKT, HML, SMB, RMW, and CMA) and the Carhart (1997) momentum factor (MOM) are shown in Table 1 (Pearson product-moment correlations below-diagonal, and Spearman rank correlations above-diagonal). The LMH portfolio returns are strongly negatively correlated with the MKT factor returns and strongly positively correlated with the HML and CMA factor returns.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Intercept	0.0185	0.0109	0.0150	0.0142	0.0103	0.0128	0.0094
	(6.48)	(3.89)	(4.74)	(4.80)	(3.98)	(4.47)	(3.75)
MKT	-0.6937						-0.4812
	(-7.04)						(-4.97)
HML		1.2319					1.0812
		(6.28)					(6.40)
SMB			-0.0502				0.3952
			(-0.28)				(2.82)
RMW				0.2714			0.1374
				(0.80)			(0.55)
CMA					1.6169		0.3026
					(8.61)		(1.29)
MOM						0.3130	0.3759
						(2.40)	(3.86)
Adj. R^2	0.1575	0.2070	-0.0011	0.0046	0.1814	0.0284	0.3569
Obs.	666	666	666	666	666	666	666

Table 2: **Regression results: LMH on the Fama-French-Carhart six factors**
This table presents the results of regressions of returns of the LMH portfolio on the MKT, HML, SMB, RMW, CMA, and MOM factors, July 1963-December 2018, 666 months. The columns labeled (1), (2), (3), (4), (5), and (6) present results for univariate specifications using only MKT, HML, SMB, MOM, RMW, and CMA, respectively, as the independent variable. The column labeled (7) presents results from the multivariate specification using all six factors as independent variables. *t*-statistics, adjusted following Newey and West (1987) using six lags, are presented in parentheses.

In Table 2, we regress the LMH portfolio returns on the Fama and French (2015) five factors and the MOM returns. Columns (1) to (6) in Table 2 show that none of these factors is able to explain the returns of the LMH portfolio, whose risk-adjusted alphas range between 1.03% and 1.85% per month, with *t*-statistics ranging from 3.89 to 6.48. Regressing the LMH portfolio returns on all the six factors' returns (column 7) indicates that the LMH portfolio has a risk-adjusted alpha of 0.94% per month with a Newey and West (1987) *t*-statistic of 3.74, adjusted for six autocorrelation lags. The returns of the LMH portfolio, therefore, are not explained by exposure to the MKT, HML, SMB, RMW, CMA, or MOM factors. We also notice the strong negative sensitivity of the LMH portfolio to the MKT factor (-0.69, with a *t*-statistic of -7.04) and the strong positive sensitivity to the HML factor (1.23, with a *t*-statistic of 6.28) and the CMA factor (1.62, with a *t*-statistic of 8.61). Finally, the sensitivity to the MOM factor is 0.31, with a significant *t*-statistic of 2.40.

Table 3 presents the results of regressions of the Fama-French-Carhart six factors on the LMH portfolio. Although in Table 2 the alpha of the LMH portfolio returns remains

	(1) MKT	(2) HML	(3) SMB	(4) RMW	(5) CMA	(6) MOM
Intercept	0.0086	0.0007	0.0022	0.0022	0.0011	0.0052
	(5.87)	(0.72)	(1.78)	(2.61)	(1.53)	(2.71)
LMH	-0.2288	0.1690	-0.0082	0.0224	0.1129	0.0954
	(-7.49)	(7.21)	(-0.28)	(0.83)	(6.88)	(2.17)
Adj. R^2	0.1575	0.2070	-0.0011	0.0046	0.1814	0.0284
Obs.	666	666	666	666	666	666

Table 3: **Regression results: Fama-French-Carhart six factors on LMH**
This table presents the results of regressions of returns of the MKT, HML, SMB, RMW, CMA, and MOM factors on the LMH portfolio, July 1963-December 2018, 666 months. t-statistics, adjusted following Newey and West (1987) using six lags, are presented in parentheses.

strongly economically and statistically significant, and is thus unexplained by any of these factors (separately or together), Table 3 indicates that, with the exception of the market factor and the momentum factor, the alphas of all other factors become either statistically insignificant (HML, SMB, CMA) or weakly economically significant (RMW). In particular, the alpha from regressing the HML factor returns on the LMH portfolio returns is 0.07% per month, and the alpha of the CMA factor is 0.11% per month. These values are small in both practical and statistical terms. We take these results as evidence that the LMH portfolio is able to price other factors, in particular HML and CMA. Put differently, following logic from Barillas and Shanken (2017), the HML and CMA factors are redundant for describing average returns, as they appear to be explained by exposure to the LMH portfolio alone.

Before testing Proposition 1, we also verify the prediction of Eq. (4), which states that assets deemed by the econometrician as being in low-demand (Δ^+ assets) have a higher β_Δ. Indeed, the correlation between β_Δ and the average of Δ over the full sample ($\bar{\Delta}$) is 54%. Regressing β_Δ on $\bar{\Delta}$ yields a positive slope of 0.083, with a t-statistic of 2.94, but not statistically different from $(\bar{\Delta}'\bar{\Delta})^{-1} = 0.12$, as Eq. (4) predicts. Consistent with Eq. (4), low-demand assets have higher β_Δ which, according to Proposition 1, should command a positive risk premium.

The following Corollary transforms Eq. (2) of Proposition 1 into an equivalent relation that is testable using standard two-stage multivariate regressions.

Corollary 1.1. *Eq.* (2) *can be equivalently written as*

$$\mu = \mu_M \widetilde{\beta} + \mu_\Delta \widetilde{\beta}_\Delta, \tag{5}$$

where $\mu_M \equiv \mu' M$*,* $\mu_\Delta \equiv \mu' \Delta$*, and* $\widetilde{\beta}$ *and* $\widetilde{\beta}_\Delta$ *are jointly estimated from multivariate time-*

series regressions of assets' excess returns on the excess returns of the market and of the LMH portfolio. Furthermore, $\widetilde{\beta}$ *and* $\widetilde{\beta}_\Delta$ *are linear combinations of* β *and* β_Δ*:*

$$\widetilde{\beta} = \frac{\sigma_\Delta^2}{\sigma_\Delta^2 \sigma_M^2 - \sigma_{M\Delta}^2}(\sigma_M^2 \beta - \sigma_{M\Delta}\beta_\Delta) \quad \textit{and} \quad \widetilde{\beta}_\Delta = \frac{\sigma_M^2}{\sigma_\Delta^2 \sigma_M^2 - \sigma_{M\Delta}^2}(\sigma_\Delta^2 \beta_\Delta - \sigma_{M\Delta}\beta), \tag{6}$$

where $\sigma_{M\Delta}$ *is the covariance between the returns of the market portfolio and the returns of the LMH portfolio.*

Proof. See Appendix A.1. □

Testing Proposition 1 is therefore equivalent to testing Eq. (5). We first regress the time series returns of each of the 25 *Size-B/M* portfolios on the excess returns of MKT and LMH. Table 4 shows the intercepts and the two factor slopes for these time-series regressions. Interestingly, none of the intercepts, which range from -0.26% to 0.21%, are statistically significant, with t-stats between -1.30 and 1.71. This is the case even for extreme growth stocks, which are a typical problem for traditional factor models (Fama and French, 2015). The slopes of the market factor are positive and strongly statistically significant, with t-stats ranging from 27.75 to 66.71. Small stocks tend to have higher betas than large stocks, whereas there is no clear pattern on the B/M dimension. The slopes on the LMH portfolio are mostly negative for the extreme growth stocks and positive for the extreme value stocks, and a large majority are statistically significant. Because the average excess return on the LMH portfolio is positive, the negative slopes of growth stocks lowers their average excess returns and the positive slopes of value stocks increases their average excess returns. Finally, the R^2 coefficients of the 25 regressions (unreported here) range from 0.60 to 0.89.

We then use the estimates from Table 4 in cross-sectional regressions of average excess returns on betas. For the sake of comparison, column (1) of Table 5 shows a direct test of the CAPM. When betas on the market are used alone, we uncover the typical failure of the CAPM: a strong positive intercept and a slope not significantly different from zero, in this case slightly negative (see also panel (b) in Figure 1). (Standard errors and t-statistics in Table 5 are corrected for cross-sectional correlations in alphas and for errors in estimating betas (Shanken, 1992).)

Column (2) presents the direct test of Corollary 1.1. When both betas on the market and on the LMH portfolio are used as explanatory variables, the slope on β_{MKT} changes sign from negative to positive, although it remains statistically insignificant (t-stat 0.91). The intercept is virtually zero, the slope on β_{LMH} is strongly statistically significant (t-stat 4.56), and the two-factor model explains 89% of the variation in average returns. The values of the two slopes are to be compared with their historical counterparts: over the

$$R_n(t) = \alpha_n + \widetilde{\beta}_n R_{MKT}(t) + \widetilde{\beta}_{\Delta,n} R_{LMH}(t) + \epsilon_n(t), \quad n = 1, ..., 25$$

	Low	2	3	4	High	Low	2	3	4	High
			$\alpha_n(\%)$					t-stat(α_n)		
Small	-0.26	0.06	0.07	0.19	0.21	-1.30	0.32	0.47	1.35	1.42
2	-0.04	0.08	0.13	0.10	0.07	-0.29	0.62	1.21	1.00	0.56
3	-0.07	0.08	0.08	0.12	0.20	-0.57	0.87	0.92	1.31	1.63
4	0.05	0.04	0.01	0.15	0.08	0.56	0.51	0.14	1.71	0.68
Big	0.07	0.06	-0.03	-0.06	0.07	1.08	1.02	-0.45	-0.61	0.53
			$\widetilde{\beta}_n$					t-stat($\widetilde{\beta}_n$)		
Small	1.32	1.25	1.15	1.11	1.18	27.75	29.33	33.63	33.42	33.38
2	1.32	1.20	1.13	1.11	1.24	36.56	40.73	42.26	43.94	38.87
3	1.27	1.17	1.06	1.06	1.16	42.10	52.54	49.44	47.69	39.45
4	1.19	1.09	1.08	1.03	1.16	51.74	60.39	54.36	49.41	40.69
Big	0.96	0.94	0.91	0.93	1.00	64.48	66.71	48.86	39.86	32.13
			$\widetilde{\beta}_{\Delta,n}$					t-stat($\widetilde{\beta}_{\Delta,n}$)		
Small	-0.12	0.04	0.06	0.14	0.16	-4.53	1.47	3.25	7.27	7.95
2	-0.11	0.03	0.09	0.13	0.16	-5.21	1.79	5.78	9.21	8.57
3	-0.07	0.06	0.06	0.12	0.12	-4.13	4.41	5.05	9.40	7.10
4	-0.05	-0.01	0.07	0.09	0.08	-3.51	-0.52	6.19	7.51	4.89
Big	-0.05	-0.02	0.06	0.04	0.03	-5.94	-2.22	5.84	2.67	1.58

Table 4: **Regressions for 25 *Size-B/M* portfolios**
This table shows two-factor intercepts, slopes for MKT and LHM, and t-statistics for these coefficients, July 1963-December 2018, 666 months. The two-factor regression equation is provided above the table. t-statistics, adjusted following Newey and West (1987) using three lags, are presented in parentheses.

period 7/1963-12/2018, the average monthly excess return on the market was 0.52% and on the LMH portfolio 1.49%. The slope on β_{MKT} (0.49%) is thus not significantly different from the historical market risk premium of 0.52% (t-stat -0.06), whereas the slope on β_{LMH} (2.41%) is marginally statistically different from 1.49% (t-stat 1.75). The Wald test that the intercept equals zero, the first slope equals 0.52%, and the second slope equals 1.49% is not rejected at 5% significance level (p-value 35%), suggesting that the two-factor model in column (2) is a reasonable description of expected returns.

Figure 2, panel (a) depicts the performance of the CAPM and panel (b) depicts the performance of the two-factor specification MKT+LMH. The vertical axis plots the unconditional expected excess returns for the 25 portfolios, whereas the horizontal axis plots the predicted values from columns (1) and (2) of Table 5. The points lie closely to a 45° line

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Intercept	0.0117	0.0005	0.0122	-0.0007	0.0098	0.0007	0.0028	0.0007
	(3.01)	(0.10)	(4.63)	(-0.18)	(3.31)	(0.17)	(0.66)	(0.18)
β_{MKT}	-0.0042	0.0049	-0.0067	0.0064	-0.0047	0.0049	0.0025	0.0048
	(-1.00)	(0.92)	(-2.16)	(1.43)	(-1.37)	(1.13)	(0.55)	(1.13)
β_{LMH}		0.0241		0.0284		0.0263		0.0264
		(4.57)		(5.56)		(6.25)		(5.96)
β_{HML}			0.0034	0.0033	0.0030	0.0032	0.0033	0.0032
			(3.09)	(2.91)	(2.70)	(2.88)	(2.98)	(2.87)
β_{SMB}			0.0015	0.0021	0.0020	0.0021	0.0021	0.0021
			(1.23)	(1.72)	(1.63)	(1.75)	(1.71)	(1.75)
β_{RMW}					0.0049	0.0012	0.0057	0.0011
					(2.88)	(0.71)	(2.83)	(0.60)
β_{CMA}					-0.0007	-0.0003	-0.0013	-0.0003
					(-0.38)	(-0.19)	(-0.63)	(-0.18)
β_{MOM}							0.0254	0.0015
							(3.47)	(0.18)
Adj. R^2	0.0479	0.8852	0.6272	0.8898	0.7270	0.8900	0.8126	0.8835
GLS R^2	0.1087	0.6457	0.2838	0.6595	0.3352	0.6781	0.4636	0.6801
Obs.	25	25	25	25	25	25	25	25

Table 5: **Cross-sectional regressions: Average excess returns on factor betas**
This table presents the results of regressions of average excess returns for 25 size and book/market sorted portfolios (Fama and French, 1993), on various combinations of seven factors: MKT, LMH, HML, SMB, RMW, CMA, and MOM, July 1963-December 2018, 666 months. t-statistics, adjusted following Shanken (1992), are presented in parentheses.

(dashed line) only in panel (b). Overall, column (2) of Table 5 and panel (b) of Figure 2 show support for Corollary 1.1: the market portfolio and the LMH portfolio explain a significant fraction of cross-sectional variation in expected returns and thus the two-factor model (5) does a good job in capturing the returns of the 25 *Size-B/M* portfolios.

Column (3) in Table 5 shows the results for the Fama and French (1993) three-factor model, whereas column (4) adds the LMH portfolio. The striking difference between columns (3) and (4) is the sign of the slope with respect to the market factor. Whereas in column (3) the estimated premium is negative and statistically significant, in column (4) the estimated premium is positive and close to the average excess market return of 0.52%. Yet, the intercept is strongly statistically significant for the Fama and French (1993) model, and the R^2 coefficient is considerably lower than in column (3). This suggests that the LMH portfolio acts like an omitted variable in column (3), distorting regression coefficients.

Column (5) shows the results for the Fama and French (2015) five-factor model, whereas column (6) adds the LMH portfolio. Once again, the market estimated premium changes

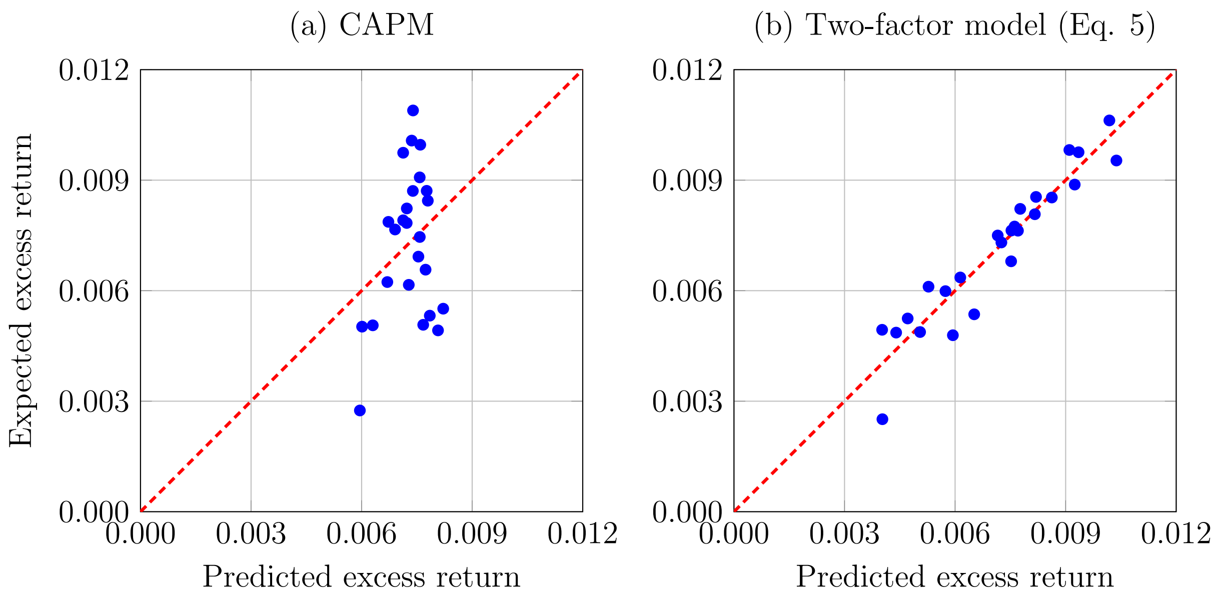

Figure 2: **Average excess returns vs. prediction**
Expected excess monthly returns for the 25 *Size-B/M* portfolios (y-axis) vs predicted monthly excess returns (x-axis) from the regression specifications in columns (1) and (2) of Table 5, July 1963-December 2018, 666 months.

sign and the alpha becomes statistically insignificant when adding LMH. Columns (7) and (8) further add the momentum factor. Curiously, when the LMH is added to the regression (column 8), the momentum factor loses significance. This suggests that the LMH portfolio may be able to explain momentum returns, a finding that we confirm in Section 4. Finally, the slope on the LMH portfolio returns is strongly statistically significant in all the specifications (t-stats ranging from 4.57 to 6.25), whereas with the exception of the HML factor, all other factors lose their statistical significance in presence of LMH.

Because in the mean-variance space high cross-sectional R^2s are not necessarily indicative of a good fit (Roll and Ross, 1994; Kandel and Stambaugh, 1995), we also report GLS R^2s, as proposed by Lewellen, Nagel, and Shanken (2010). GLS R^2s measure the proximity of a given model's portfolio to the minimum-variance frontier (mean-variance efficiency is obtained when the GLS R^2 is one). As Table 5 shows, according to this metric, the two-factor model of column (2) is the closest to mean-variance efficiency (GLS R^2 0.65) when compared to the Fama and French (1993) three-factor model (GLS R^2 0.28), to the Fama and French (2015) five-factor model (GLS R^2 0.34), and the Fama and French (2015)-Carhart (1997) six factor model (GLS R^2 0.46).

In sum, we have shown theoretically that, when the CAPM fails, a low-minus-high portfolio explains, together with the market, 100% of the cross-sectional variation in returns.

We have confirmed empirically these properties of the LMH portfolio within the Fama and French (1993) *Size-B/M* portfolio space. We emphasize, however, that this factor appears significant *regardless* the reason behind the CAPM rejection. In particular, if the CAPM is rejected by mistake—because the econometrician does not use the correct market portfolio (Roll, 1977), or does not have the same information as investors do (Andrei et al., 2018)—then the LMH portfolio becomes a proxy for this mistake. In this case, not only the econometrician will reject the CAPM, but will also fail to reject a multifactor model. In Section 3, we will illustrate this possibility in an equilibrium model of stock returns.

Another implication of the above results is that many investment strategies can proxy for the LMH portfolio. In other words, the existence of the LMH portfolio opens the door to a myriad of profitable investment strategies, or "smart beta" funds, as we show next.

2.1 "Smart Beta" Funds

When the CAPM does not hold, the econometrician can construct an infinity of long-short portfolios whose returns are not explained by the market. When one such portfolio is added to the market portfolio, the combined portfolios can be interpreted as a "smart beta" fund. To see this, consider a zero investment long-short portfolio A, and let the covariance of returns of this portfolio with the returns of the market portfolio, σ_{AM}, be zero (a market-neutral portfolio). The following Lemma, which results directly from multiplying Eq. (20) of Corollary 3.3 with A', shows that this portfolio earns a non-zero alpha as long as its returns covary with the returns of the LMH portfolio.

Lemma 1. *Let the covariance of A's returns with the returns of the LMH portfolio be $\sigma_{A\Delta}$ and assume, without loss of generality, that $\sigma_{A\Delta} \geq 0$ (in case of the contrary, we would consider the portfolio $-A$). The expected return on the portfolio A is given by*

$$\mu_A = \frac{\mu_T}{\sigma_T^2}\sigma_{A\Delta} \geq 0. \tag{7}$$

Since we have assumed that A is market neutral, the expected return μ_A is the CAPM alpha of portfolio A. Consider now the portfolio $M+A$. According to Lemma 1, the expected return of this portfolio is $\mu_M + \mu_A \geq \mu_M$. Yet, its covariance with the market portfolio equals σ_M^2, and thus its beta is always one. As a result, this portfolio earns returns in excess of the market despite having a beta of one. This makes it a smart beta fund.

Can we locate smart beta funds in the mean-variance framework? Most importantly, how many such smart beta funds can we find? As we show in the Proposition below, we can actually draw an *efficient frontier* of smart beta funds—the set of smart beta funds that

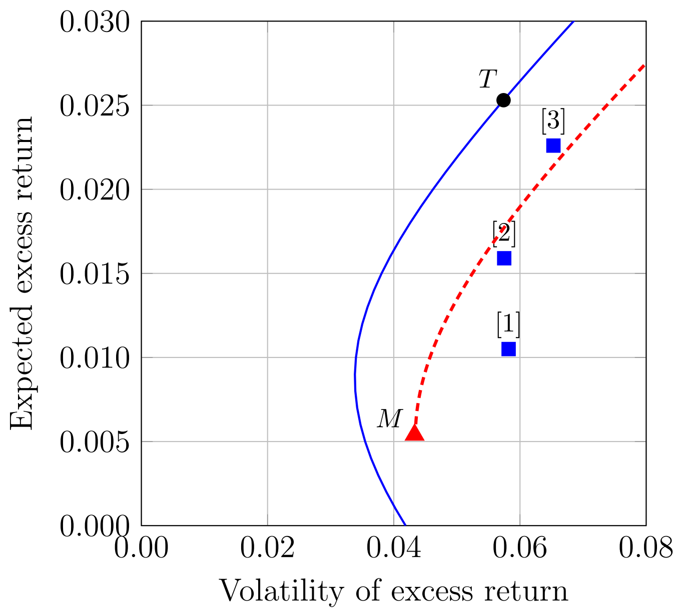

Figure 3: **The set of "smart beta" funds**
The solid line is the full-sample minimum-variance frontier for 25 size and book/market sorted portfolios (Fama and French, 1993), July 1963-December 2018, 666 months. The dashed line represents the efficient set of smart beta funds: portfolios that have a beta of one with the market, but an expected return higher than the expected return of the market. The squares labeled [1], [2], and [3] are defined in the text.

have minimum volatility of returns for a given expected excess return.

Proposition 2. *There exists a minimum-variance efficient set of smart beta funds for all expected returns $\mu_{SB} \geq \mu_M$. The portfolio composition of the smart beta fund with expected return μ_{SB}, SB, is given by*

$$SB = \zeta_1 \Sigma^{-1} \mathbf{1}_N + \zeta_2 \Sigma^{-1} \mu + \zeta_3 M, \tag{8}$$

where ζ_1, ζ_2, and ζ_3 are affine functions of μ_{SB}, defined in Appendix A.2. The variance of its returns is a quadratic function of μ_{SB}:

$$\sigma^2_{SB} = \zeta_1 + \zeta_2 \mu_{SB} + \zeta_3 \sigma^2_M. \tag{9}$$

Proof. See Appendix A.2. □

The smart beta efficient frontier is a hyperbola in the mean-standard deviation space. Its global minimum-variance portfolio is the market portfolio M (in other words, M is the smart beta fund with an alpha of zero). All the portfolios on the upper limb of the hyperbola are then funds with positive alphas. Figure 3 depicts the smart beta efficient set in the Fama and French (1993) 25 *Size-B/M* portfolio space. The solid line represents the mean-variance efficient frontier, whereas the dashed line represents the smart beta efficient frontier. This line lies inside the global minimum-variance frontier (intuitively, smart beta funds are constrained to have a beta of one with the market, and this additional constraint pushes their efficient set inside). The squares labeled [1], [2], and [3] are the same as in panel (a) of Figure 1 and represent three investment strategies: [1] is the Fama and French (1993) 3-factor investment strategy (that is, MKT plus HML plus SMB), [2] is the Fama and French (2015) 5-factor investment strategy, and [3] is the Fama and French (2015)-Carhart (1997) 5-factor-plus-momentum investment strategy. These investment strategies, and in particular [2] and [3], are close to the smart beta efficient frontier.

The fact that these well-known investment strategies are close to the (ex-post) smart beta efficient set shows that academic research has successfully managed to build better models for the cross section of returns. But, Figure 3 also suggests an uncomfortable implication: there are in fact an infinity of smart beta funds, which can lead to a never-ending quest for identifying pricing anomalies in the cross section of returns.

3 An Equilibrium Interpretation of the LMH Portfolio

The aim of this section is to characterize the LMH portfolio in an equilibrium model. We intentionally build a model in which a true CAPM relationship holds for investors, but it fails for the econometrician (Andrei et al., 2018). In this model, there is no economic reason for firms with low market-to-book ratios and for firms with low investment rates to appear relatively riskier than the CAPM predicts. Yet, as we elaborate below, the econometrician concludes that these firms command an additional risk premium, and consequently rejects the CAPM and fails to reject a multifactor model of returns.

Consider a one-period economy in which the market consists of one risk-free asset with gross return normalized to 1 and N firms indexed by $n = 1, ..., N$. We assume the value of assets in place to be the same for all firms, and we denote this value by K. Firms are heterogeneous with respect to the productivity of their assets. Specifically, firm productivities are unobservable at time 0 and have a common factor structure:

$$Z = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{bmatrix} (F + \widetilde{F}) + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_N \end{bmatrix} \equiv \Phi(F + \widetilde{F}) + \epsilon. \tag{10}$$

The common productivity shock $\widetilde{F}$ and each firm-specific shock ϵ_j are independently normally distributed with means 0 and precisions τ_F and τ_ϵ. Without loss of generality, we assume that the cross-sectional average of firms' loadings on the common productivity shock is positive: $\bar{\Phi} \equiv N^{-1} \sum_{n=1}^{N} \phi_n > 0$.

The final values of firms at time 1 depend on their assets in place and their productivities:

$$D = KZ. \tag{11}$$

We assume that N claims on these final values (one for each firm) are traded in financial markets. The economy is populated by a continuum of investors indexed by $i \in [0, 1]$, who choose their portfolio at time 0 and derive utility from terminal wealth with constant absolute risk aversion coefficient γ:

$$\begin{aligned} &\max_{\omega_i} \mathbb{E}\left[-e^{-\gamma W_i} | \mathcal{F}_i\right] \\ \text{s.t.} \quad &W_i = W_{i,0} R_f + \omega_i'(D - P R_f), \end{aligned} \tag{12}$$

where $W_{i,0}$ is investor i' initial wealth, ω_i is investor i's portfolio (in units of assets), $R_f \geq 1$ is the gross interest rate, P is the vector of equilibrium prices, and $\mathcal{F}_i$ is the information set of investor i (to be described below). Without loss of generality, we fix $W_{i,0} = 0$ and $R_f = 1$. Finally, we define excess returns as $R \equiv D - P R_f$.

Investors know the structure of realized payoffs in Eq. (10)-(11), but do not observe the common productivity shock $\widetilde{F}$ and firms' specific productivity shocks ϵ. Each investor i forms expectations about $\widetilde{F}$ based on both a private signal $V_i = \widetilde{F} + v_i$ and a public signal $G = \widetilde{F} + v$. The signal noises v and $v_i \perp v$, $\forall i$, are unbiased and independently normally distributed with precisions τ_G and τ_v, respectively. These signals, together with the vector of prices, comprise the information set of investor i, $\mathcal{F}_i = \{V_i, G, P\}$.

Conditional on the information set $\mathcal{F}_i$, each investor builds expectations about excess returns for all firms. Defining $\widehat{\mu}_i \equiv \mathbb{E}[R|\mathcal{F}_i]$ as the vector of expected returns and $\widehat{\Sigma} \equiv \text{Var}[R|\mathcal{F}_i]$ as the conditional covariance matrix of returns (which is identical across investors), investor i's optimal portfolio choice is

$$\omega_i = \frac{1}{\gamma} \widehat{\Sigma}^{-1} \widehat{\mu}_i. \tag{13}$$

The total supply of shares is noisy and unobservable (Grossman and Stiglitz, 1980). It equals $M + \widetilde{M}$, where we denote by M the *unconditional* market portfolio and by $\widetilde{M}$ the noise in supplies. The unconditional market portfolio M is a vector with strictly positive values that sum up to one, $M = [M_1 \ ... \ M_N]'$, whereas $\widetilde{M}$ is a vector whose elements are normally and independently distributed with precision τ_M. Market clearing yields

$$\int_i \omega_i di = M + \widetilde{M}. \tag{14}$$

Proposition 3 characterizes equilibrium prices in this economy.

Proposition 3. *There exists a partially revealing rational expectations equilibrium in which the vector of market-to-book ratios, P/K, is given by*

$$\frac{P}{K} = \Phi F + \xi_0 M + \alpha \widetilde{F} + gG + \xi \widetilde{M}, \tag{15}$$

where the coefficients ξ_0 $(N \times N)$, α $(N \times 1)$, g $(N \times 1)$, and ξ $(N \times N)$ solve

$$\xi_0 = -\gamma \frac{\widehat{\Sigma}}{K}, \quad \alpha = \Phi \frac{\tau - \tau_F - \tau_G}{\tau}, \quad g = \Phi \frac{\tau_G}{\tau}, \quad \xi = -\frac{\gamma K + \sqrt{\tau_M \tau_P}}{\tau} \Phi \Phi' - \frac{\gamma K}{\tau_\epsilon} \mathbb{I}_N, \tag{16}$$

and the parameter $\tau \equiv Var^{-1}[\widetilde{F} | \mathcal{F}_i]$ is a scalar determined in equilibrium as the unique positive solution of a cubic equation.

Proof. See Appendix A.3. □

Consider a fictitious investor who holds the average beliefs $\widehat{\mu} = \int_i \widehat{\mu}_i di$ and $\widehat{\Sigma} = \mathrm{Var}[R|\mathcal{F}_i]$. Corollary 3.1 results directly from Proposition 3 and describes the unconditional CAPM relation that holds from the point of view of this investor.

Corollary 3.1. *In this economy, an unconditional CAPM relation holds:*

$$\mu = \frac{\widehat{\Sigma} M}{\widehat{\sigma}_M^2} \mu_M = \widehat{\beta}\, \mu_M, \tag{17}$$

where $\mu \equiv \mathbb{E}[\widehat{\mu}]$, $\widehat{\sigma}_M^2 \equiv M' \widehat{\Sigma} M$ is the variance of excess returns for the unconditional market portfolio, and $\mu_M \equiv M' \mu$ is the unconditional expected excess return on the market.

Proof. Write the market clearing condition (14) as $\widehat{\mu} = \gamma \widehat{\Sigma}(M + \widetilde{M})$ and take unconditional expectation to obtain $\mu = \gamma \widehat{\Sigma} M$. Multiplication with M' on both sides yields $\mu_M = \gamma \widehat{\sigma}_M^2$. Then divide $\mu = \gamma \widehat{\Sigma} M$ by $\mu_M = \gamma \widehat{\sigma}_M^2$ to obtain (17). □

The vector $\widehat{\beta}$ represents the *true* betas in this economy, which are obtained under the average agent's information set. An equivalent way of stating Corollary 3.1 is that the unconditional market portfolio M is mean-variance efficient and thus commands the highest Sharpe ratio in the economy (Roll, 1977).

Corollary 3.2. *Under the information set of the average agent, the Sharpe ratio of the market portfolio reaches its maximum attainable level in the economy:*

$$\frac{\mu_M}{\widehat{\sigma}_M} = \sqrt{\mu' \widehat{\Sigma}^{-1} \mu}. \tag{18}$$

Proof. The proof follows from efficient set mathematics. Define $\widehat{B} \equiv \mathbf{1}' \widehat{\Sigma}^{-1} \mu$, $\widehat{C} \equiv \mu' \widehat{\Sigma}^{-1} \mu$. The tangency (market) portfolio has an expected excess return of $\mu_M = \widehat{C}/\widehat{B}$ and a variance of excess returns of $\widehat{\sigma}_M^2 = \widehat{C}/\widehat{B}^2$. Thus, $\mu_M/\widehat{\sigma}_M = \sqrt{\widehat{C}}$, which yields (18). □

The econometrician observes realized returns on all assets and on the market portfolio M. The law of iterated expectations implies that the econometrician correctly measures μ and μ_M. But, because econometrician's information set is coarser than the average agent's set, the law of total variance implies that the covariance matrix of excess returns of the econometrician, Σ, is different that $\widehat{\Sigma}$. As a result, the econometrician rejects the CAPM.

Corollary 3.3. *Under the information set of the econometrician, the unconditional market portfolio is not mean-variance efficient:*

$$\frac{\mu_M}{\sigma_M} < \sqrt{\mu' \Sigma^{-1} \mu}, \tag{19}$$

where Σ is the unconditional covariance matrix of realized excess returns and $\sigma_M \equiv \sqrt{M' \Sigma M}$ is the volatility of excess returns of the market portfolio. Thus, the econometrician rejects the CAPM.

Proof. See Appendix A.4. □

For the econometrician, there will always be some portfolio which is ex-post efficient (Roll, 1977). According to Corollary 3.3, this portfolio is different from the true market portfolio M. Thus, as in Proposition 1, the econometrician can build a LMH portfolio that, together with the market portfolio, explains 100% of the cross-sectional variation in excess returns.

Corollary 3.4. *For the econometrician, the portfolio $T = \Sigma^{-1}\mu/B$, where $B \equiv \mathbf{1}' \Sigma^{-1} \mu$, is ex-post efficient. Since $T \neq M$ (Corollary 3.3), the econometrician can define $\Delta \equiv T - M$ and write the expected returns on all assets as*

$$\mu = \frac{\mu_T \sigma_M^2}{\sigma_T^2} \beta + \frac{\mu_T \sigma_\Delta^2}{\sigma_T^2} \beta_\Delta, \tag{20}$$

where $\mu_T \equiv \mu' T$, $\sigma_T^2 \equiv T' \Sigma T$, $\sigma_\Delta^2 \equiv \Delta' \Sigma \Delta$, $\beta \equiv \Sigma M / \sigma_M^2$, and $\beta_\Delta \equiv \Sigma \Delta / \sigma_\Delta^2$.

Proof. The proof follows the same steps as the proof of Proposition 1. □

The set of market betas that the econometrician computes, β, are based on the covariance matrix of realized excess returns Σ and thus are different than the set of true betas, which are based on $\widehat{\Sigma}$. Given this, Corollary 3.4 has two main implications. First, the econometrician rejects the CAPM (Andrei et al., 2018). Second, the econometrician can build two sets of betas, one with respect to the market portfolio and one with respect to the LMH portfolio, which explain ex-post 100% of the variation in expected excess returns.

We have therefore uncovered the same result as in Proposition 1. When the econometrician rejects the CAPM, there will always be a portfolio Δ that, when added as a second factor, helps explain the entire cross-sectional variation in excess returns. The question we now address is, to what extent an econometrician can proxy for this portfolio by using publicly available information?

3.1 The LMH Portfolio and the Value Factor

We will now provide a theoretical link between the portfolio Δ constructed in Corollary 3.4 and the value factor. We directly state the main result in the following Proposition.

Proposition 4. *In this economy, the cross-sectional variation in excess returns is entirely explained by firms' betas together with the vector of market-to-book ratios, $\mathbb{E}[P]/K$:*

$$\mu = \lambda_1 \beta + \lambda_2 \frac{\mathbb{E}[P]}{K}, \text{ with } \lambda_1 > 0 \text{ and } \lambda_2 < 0. \tag{21}$$

Proof. See Appendix A.5. □

The negative sign of λ_2 admits the following interpretation. Firms with low market-to-book ratios (value firms) have higher expected excess returns after controlling for market beta. Because we have not assumed ex-ante that these firms are inherently riskier, in this economy a value premium cannot possibly be reward for fundamental risk. Rather, the value premium simply reflects mis-measurement in beta estimates.

In the same way, the fact that value firms command a risk premium that is not explained by exposure to the market cannot be regarded as evidence against the CAPM. Corollary 3.1 shows that a true CAPM holds in this economy. Yet, because the econometrician operates under a coarser information set than economic agents who collectively hold the market portfolio, mis-measurement in betas not only leads to the rejection of the CAPM, but also creates a value factor.

In the data, the returns on the LMH portfolio and on the HML factor have a strong positive correlation of 0.46 (Table 1); regressing the HML factor on the LMH portfolio yields a zero intercept and a slope that is strongly statistically significant (Table 3). Based on these results, one can perhaps say that the HML factor is a proxy for the LMH portfolio that we have built in Section 2.

3.2 The LMH Portfolio and the Investment Factor

The q theory of investment (Jorgenson, 1963; Tobin, 1969; Lucas and Prescott, 1971; Hayashi, 1982) predicts a strong relationship between firms' market values and their investment rates. Because firms' valuations are driven by expectations of their future cash-flows, high valuations must indicate profitable opportunities and therefore highly-valued firms should invest more aggressively. Recent data lend support for this positive relationship (Andrei, Mann, and Moyen, 2018).

In light of the results of the previous section, factors that are built based on firms' investment rates must then correlate with the LMH portfolio. One such factor is the "Conservative-Minus-Aggressive" (CMA) factor, whose returns represent the difference between the returns on diversified portfolios of the stocks of low and high investment firms. Fama and French (2015) document that the CMA factor is surprisingly strong in explaining the cross-section of returns, to the extent that the HML factor becomes redundant when CMA is included in regressions. Indeed, as can be seen from Table 1, the correlation coefficient between HML and CMA is 0.70.

In this section, we provide a theoretical link between the LMH portfolio and the CMA factor. Our argument is heuristic, in that we consider a minimal extension of our setup in which market equilibrium is reached as in Proposition 3, but we incorporate firms' decision to invest. More precisely, let the ex-post profit of a firm n be

$$\Pi_n = \left[\phi_n(F + \widetilde{F}) + \epsilon_n\right](K + I_n) - I_n - \frac{a}{2}\left(\frac{I_n}{K}\right)^2 K, \tag{22}$$

where I_n represents the investment decision of firm n. The last term represents adjustment costs, which are strictly convex ($a > 0$) and linear homogeneous in I and K (Hayashi, 1982).

We assume that the firms-specific component, ϵ_n, is perfectly observed by the insider of the firm (hereafter "the manager"). Furthermore, the manager observes a private signal about $\widetilde{F}$, $V_m = \widetilde{F} + v_m$, where $v_m \sim N(0, 1/\tau_{v,m})$. Because this is an imperfect signal, the manager also uses public prices to learn about $\widetilde{F}$, as investors do. Maximization of (22) yields the optimal investment decision

$$\frac{I_n^*}{K} = -\frac{1}{a} + \frac{1}{a}\left(\phi_n\, \mathbb{E}[F + \widetilde{F}|\mathcal{F}_m] + \epsilon_n\right), \tag{23}$$

where $\mathcal{F}_m$ is the information set of the manager. This yields a direct relationship between the investment rate of firm n and the beliefs of the manager. Taking unconditional expectation and writing this relationship for all N firms yields

$$\frac{\mathbb{E}[I^*]}{K} = -\frac{1}{a} + \frac{1}{a}\Phi F. \tag{24}$$

Eq. (24) provides a direct link between firms' average investment rates and firms' exposure to the common factor. Let us assume, for the sake of the argument, that equilibrium prices preserve the form given in Proposition 3.[6] The following Corollary, then, results directly from Proposition 4 and Eq. (24).

Corollary 4.1. *In this economy, the cross-sectional variation in excess returns is entirely explained by firms' betas together with the vector of investment ratios,* $\mathbb{E}[I^*]/K$*:*

$$\mu = \eta_0 + \eta_1\beta + \eta_2\frac{\mathbb{E}[I^*]}{K}, \text{ with } \eta_0 < 0, \ \eta_1 > 0 \text{ and } \eta_2 < 0. \tag{25}$$

Proof. See Appendix A.6 □

Firms with low investment rates (conservative firms) have higher returns after controlling for market beta. The q theory of investment implies that these firms also have low valuations, which, in the context of our model, means low market-to-book rates. Because these firms earn a positive risk premium after controlling for their market beta (Proposition 4), it implies that a CMA factor must command a positive risk premium, hence $\eta_2 < 0$.

Thus, firms with lower investment rates earn a risk premium beyond the premium required by their exposure to the market. Our model therefore predicts a positive relationship between the returns of the LMH portfolio and the returns of the CMA factor. Indeed, in column (5) of Table 3 we regress the CMA factor on the LMH portfolio and obtain a zero intercept (0.0011, with t-stat 1.52) and a strongly positive slope (0.1129, t-stat 6.87). Yet, the fact that low investment firms command a risk premium that is not explained by exposure to the market cannot be regarded as evidence against the CAPM. Since in this economy a true CAPM holds (Corollary 3.1), the investment premium may represent mis-measurement in beta estimates.

[6] We do not revisit here the equilibrium of Proposition 3. Because the manager has information about $\widetilde{F}$ and ϵ_n, the investment decision (23) is public information from which investors can learn. This would require the addition of N public signals to the learning problem of each agent. Furthermore, the optimal investment decision will introduce quadratic terms in the firms' final payoffs, which breaks the linearity of the CARA-normal setup. Overcoming this technical issue would require a different model (Albagli, Hellwig, and Tsyvinski, 2011a,b; David, Hopenhayn, and Venkateswaran, 2016), which is beyond the scope of our paper. Alternatively, one can focus on first-order terms in firms' payoffs (Bai, Philippon, and Savov, 2016), which would preserve the linearity of the model.

3.3 The Factor Zoo

Beta mis-measurement creates the illusion that factors other than the market are asset-pricing relevant. In the context of Proposition 4, there is only one such additional factor because the covariance matrix $\widehat{\Sigma}$ only has two distinct eigenvalues; this is equivalent to saying that asset payoffs are driven by one common factor. Suppose that we allow payoffs to be driven by $K \leq N$ common factors. The empiricist may now uncover up to as many additional factors as there are assets in the market, thus unleashing the factor zoo.

To link the LMH portfolio to the factor zoo, recall that there always exists a portfolio that is ex-post efficient (Roll, 1977). A restatement of this observation is that adding LMH formed ex-post to the CAPM relation always produces an R^2 of 1. But then the K additional factors can be identified as principal components of LMH formed ex-post. We now examine this idea in the context of a model with K factors driving payoffs.

Denote a vector of $K \leq N$ independent factors by $\mathbf{F} \equiv [\ F_1 \quad F_2 \quad \cdots \quad F_K\ ]'$. Let this vector be normally distributed with mean F and covariance matrix $\tau_F^{-1}\mathbf{I}_K$. Let the vectors of realized asset payoffs have the structure:

$$D = \Phi\mathbf{F} + \epsilon, \tag{26}$$

where we decompose the matrix of loadings Φ as:

$$\Phi \equiv \begin{bmatrix} \Phi_1 & \Phi_2 & \ldots & \Phi_K \end{bmatrix}. \tag{27}$$

That is, the vector Φ_k contains the loading of each stock on the $k-$th factor. Each investor i observes a vector of private signals about the K factors

$$\mathbf{V}^i = \mathbf{F} + v^i, \quad v^i \sim \mathscr{N}\left(\mathbf{0}, \tau_v^{-1}\mathbf{I}_K\right), \tag{28}$$

as well as a common public signal

$$\mathbf{G} = \mathbf{F} + v, \quad v \sim \mathscr{N}\left(\mathbf{0}, \tau_G^{-1}\mathbf{I}_K\right). \tag{29}$$

Other than allowing multiple factors to affect payoffs, we keep the structure of the model unchanged.

Proposition 3 characterizes equilibrium prices in this economy.

Proposition 5. *There exists a partially revealing rational expectations equilibrium in which* *the vector of stock prices is given by*

$$P = \Phi F - \gamma\widehat{\Sigma}M + \alpha\mathbf{F} + \tau_G\Phi\tau^{-1}\mathbf{G} + \beta\widetilde{M}\,, \tag{30}$$

where the coefficients α $(N \times 1)$, *and* ξ $(N \times N)$ *solve*

$$\alpha = \Phi\tau^{-1}(\tau - \tau_F\mathbf{I}_K - \tau_G\mathbf{I}_K), \quad \beta = -\sqrt{\tau_M}\Phi\tau^{-1}\tau_P'\Phi' - \gamma\left(\Phi\tau^{-1}\Phi' + \frac{1}{\tau_\epsilon}\mathbf{I}_N\right), \tag{31}$$

and the $K \times K-$*matrix* $\tau \equiv Var^{-1}[\widetilde{F}|\mathcal{F}_i]$ *denotes total precision on the vector of* $K-$*factors:*

$$\tau \equiv (\tau_F + \tau_v + \tau_G)\mathbf{I}_K + \tau_P\Phi'\Phi\tau_P, \tag{32}$$

where the $K \times K-$*matrix solves the matrix equation:*

$$\tau^{-1}\Phi'\Phi\tau_P = \sqrt{\tau_M}\frac{\tau_v}{\gamma}\left(\tau^{-1} - (\tau + \tau_\epsilon\Phi'\Phi)^{-1}\right). \tag{33}$$

Proof. See Appendix A.7. □

Importantly, in the presence of multiple factors the CAPM still holds from the perspective of the average investor. The proof follows along the lines of that of Corollary 3.1. We proceed directly with the empiricist's perception of the CAPM relation, which we characterize in the proposition below.

Proposition 6. ***(Factor Zoo).*** *In the eyes of the empiricist, expected returns satisfy the equilibrium relation:*

$$\mu = \underbrace{\frac{\sigma_M^2}{\widehat{\sigma}_M^2}\left(1 + \frac{\gamma^2}{\tau_M\tau_\epsilon}\right)^{-1}\beta\mu_M}_{\text{distorted CAPM relation}} - \underbrace{\frac{\gamma\tau_M\tau_\epsilon}{\gamma^2 + \tau_M\tau_\epsilon}\sum_{k=1}^{K}\left(\sum_{j=1}^{K}\phi_j c_{kj}\right)\Phi_k}_{K\text{ additional factors}}, \tag{34}$$

where $\phi_j \equiv \Phi_j'M$ *denotes the average loading on factor* j *and the coefficients* c_{jk} *are defined in the appendix. In this context, the beta on the LMH portfolio satisfies the relation:*

$$\beta_\Delta = \frac{\sigma_T^2\sigma_\Delta^2}{\mu_T}\frac{\gamma\tau_M\tau_\epsilon}{\gamma^2 + \tau_M\tau_\epsilon}\sum_{k=1}^{K}B_k\left(\sum_{j=1}^{K}\phi_j c_{kj}\right)\Sigma\left(M - \Sigma^{-1}\Phi_k/B_k\right), \tag{35}$$

where $B_k \equiv \Phi_k'\Sigma^{-1}\mathbf{1}$ *and thus* $\Sigma^{-1}\Phi_k/B_k$ *is the efficient portfolio fully invested in Factor* k.

Proof. See Appendix A.8. □

Eq. (34) shows how beta mis-measurement opens the gate to the factor zoo. Although none of the K factors is relevant for asset-pricing tests—the CAPM holds—the empiricist concludes that all K factors should be added to the (distorted) CAPM relation. In other words, although some factors may appear more relevant than others, they each separately increase the R squared of the relation. In fact, if we added them all, the fit would be perfect.

What each factor does is to correct the market portfolio to bring it closer to the empiricist's perceived tangency portfolio. Eq. (35) shows how. Recall that the LMH portfolio, Δ, represents the gap between the market portfolio and the perceived tangency portfolio. If these two portfolios coincided, the CAPM would no longer be rejected. Suppose the empiricist has successfully identified the K factors driving payoffs. Based on mean-variance analysis, she could form K efficient portfolios, each fully invested in each factor. Formally, the mathematics of the efficient frontier imply that the efficient portfolio invested in the $k-$th factor would take the form, $\Sigma^{-1}\Phi_k/B_k$. The portfolio LMH compares each of these efficient portfolios to the market portfolio (the term in bracket in Eq. (35)) and weighs the difference between the two according to the importance of each factor k (i.e., its average loading ϕ_k). The result is that adding a factor brings us closer to the perceived tangency portfolio.

As a final note, it is worth mentioning that the LMH portfolio takes us to the empirical tangency portfolio in one step. However, adding one factor after another would get us gradually toward the tangency portfolio. We have therefore created a fertile playing field for empirical asset pricing.

4 Further Empirical Results

In this Section, we will assess the ability of the the LMH portfolio to price additional portfolios sorts, such as portfolios based on past performance, on alternative price multiples, and on investment.

An interesting result found in Section 2 was the relatively high positive correlation between the returns of the LMH portfolio and the returns of the momentum factor (0.17, Table 1). Although the LMH portfolio did not entirely eliminate the alpha of the momentum factor (Table 3, column 6), it did eliminate its statistical significance in cross-sectional regressions (Table 5). This suggests that the LMH portfolio might be able to price stocks sorted based on past performance.

Table 6 confirms. In columns (1)-(3) of panel A, we present cross-sectional regressions on 10 portfolios sorted based on their past performance (portfolio sorts available from Professor Ken French's website). The LMH portfolio explains, together with the market, 95% of

Panel A: Sorts involving prior returns

	Momentum			ST Reversal			LT Reversal		
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Intercept	0.0170	-0.0035	0.0251	0.0022	-0.0048	-0.0003	0.0052	0.0035	0.0195
	(4.75)	(-0.42)	(3.17)	(0.68)	(-0.81)	(-0.03)	(1.68)	(0.97)	(1.47)
MKT	-0.0101	0.0114	-0.0169	0.0030	0.0099	0.0049	0.0010	0.0023	-0.0136
	(-2.55)	(1.32)	(-1.96)	(0.82)	(1.63)	(0.57)	(0.29)	(0.60)	(-1.04)
LMH		0.1058			0.0442			0.0124	
		(2.22)			(2.15)			(1.62)	
HML			-0.0093			0.0200			-0.0016
			(-1.44)			(2.20)			(-0.40)
SMB			0.0036			-0.0011			0.0064
			(0.61)			(-0.23)			(1.29)
Adj. R^2	0.1817	0.9502	0.8351	-0.0251	0.1233	0.7987	-0.1076	0.7976	0.9063
Obs.	10	10	10	10	10	10	10	10	10

Panel B: Sorts involving Investment, E/P, and CF/P

	Investment			Earnings/Price			Cashflow/Price		
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Intercept	0.0096	-0.0042	0.0104	0.0102	-0.0006	0.0088	0.0114	-0.0009	0.0115
	(3.59)	(-0.73)	(1.54)	(2.27)	(-0.14)	(0.99)	(2.47)	(-0.17)	(1.54)
MKT	-0.0037	0.0096	-0.0048	-0.0042	0.0064	-0.0031	-0.0055	0.0067	-0.0058
	(-1.19)	(1.61)	(-0.70)	(-0.88)	(1.29)	(-0.35)	(-1.14)	(1.17)	(-0.76)
LMH		0.0441			0.0239			0.0252	
		(2.59)			(2.66)			(2.19)	
HML			0.0028			0.0023			0.0012
			(1.31)			(1.44)			(0.72)
SMB			0.0038			0.0071			0.0087
			(1.27)			(1.56)			(2.06)
Adj. R^2	0.0784	0.6816	0.4318	-0.0532	0.9071	0.8698	0.0585	0.6843	0.8627
Obs.	10	10	10	10	10	10	10	10	10

Table 6: **Regression results: Various portfolio sorts**
This table presents the results of regressions of average excess returns for six different portfolio sorts on various combinations of four factors: MKT, LMH, HML, and SMB, July 1963-December 2018, 666 months. t-statistics, adjusted following Shanken (1992), are presented in parentheses.

the cross-section of returns. Furthermore, its estimated premium is positive and statistically significant (0.1058, with a t-stat of 2.22), and the intercept of the regression is not statistically different from zero. In contrast, the intercept of the Fama and French (1993) three-factor model is economically large and strongly statistical significant (0.0251, with a t-stat of 3.17), and none of the three factors are statistically significant. Overall, columns (1)-(3) of Table 6 attest to the ability of the LMH portfolio to price stocks sorted based on momentum.

Columns (4)-(9) of panel A further show results when stocks are sorted based on short-term reversal and long-term reversal. While in both cases the CAPM fares particularly bad, the LMH portfolio earns a positive risk premium, statistically significant only in one case. The Fama and French (1993) three-factor model fares relatively better in these portfolio sorts. Finally, as shown in panel B, the LMH portfolio performs extremely well in sorts based on investment, earnings/price ratios, or cashflow/price ratios. In particular, we notice that the intercepts are all statistically indistinguishable from zero when the MKT and LMH are both used in the regressions.

To summarize, the powerful explanatory power of the LMH portfolio in the 25 *Size-B/M* portfolio set transfers well in other portfolio sorts. But, as the results above show, the LMH portfolio is hardly a panacea. We emphasize, nevertheless, that the LMH portfolio need not be built exclusively from the 25 *Size-B/M* portfolio set. One can in fact build one LMH portfolio for each one of the portfolio sorts considered here. Any combination of these portfolios—always a zero-investment portfolio—would yield a global LMH portfolio.

5 Conclusion

Empirical asset pricing has identified hundreds of anomalies, and interpreted this as evidence against the CAPM. We agree that the sheer number of these anomalies makes the case against theory compelling. Yet, we argue that anomalies are not, by themselves, evidence that the theory is wrong. On the contrary, their large number could be the Achilles heel of the case against theory.

Regardless whether the CAPM is rejected for valid reasons or by mistake, finding anomalies is the unavoidable symptom of the rejection. Theoretically, one can always build a long-short portfolio that explains, together with the market, the cross section of returns. As soon as factors based on firm characteristics or on macroeconomic fundamentals covary with this portfolio, they become anomalies, although in reality they need not be. One can think of this situation as a statistical mirage that compels empiricists to accept multifactor models of returns, and theorists to struggle interpreting these models.

More significantly, though, anomalies are silent about the true cause of the CAPM rejection. Because the potential number of anomalies is unlimited, multifactor models of returns do not reveal anything meaningful about the true reasons for the difference between the theory and the data. In our theoretical model, for instance, the CAPM is rejected by mistake. But even if the CAPM is rejected for a valid reason, finding anomalies will likely not identify this reason. Anomalies are, at best, uninformative and, at worst, misleading.

Faced with such uncertainty about the true interpretation of anomalies, "the safest course may be to assume that the theory is correct." (Black and Scholes, 1974, p. 405)

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A Appendix

A.1 Proof of Corollary 1.1

Consider the following multivariate time-series relation (stacked for all assets):

$$R = \widetilde{\alpha} + \widetilde{\beta} R_M + \widetilde{\beta}_\Delta R_\Delta + u, \tag{A.1}$$

where R is the vector of excess returns on all assets, R_M is the excess return on the market, and R_Δ is the excess return on the Δ portfolio. Let the univariate time-series relation be

$$R = \alpha + \beta R_M + \varepsilon, \tag{A.2}$$

where the coefficient β is the same as in Proposition 1. Comparing Eqs. (A.1) and (A.2), there is an omitted variable (R_Δ) in Eq. (A.2). This yields

$$\beta = \widetilde{\beta} + \frac{\sigma_{M\Delta}}{\sigma_M^2} \widetilde{\beta}_\Delta. \tag{A.3}$$

By the same logic, writing $R = \alpha_\Delta + \beta_\Delta R_M + \varepsilon_\Delta$ and comparing with Eq. (A.1) yields

$$\beta_\Delta = \widetilde{\beta}_\Delta + \frac{\sigma_{M\Delta}}{\sigma_\Delta^2} \widetilde{\beta}. \tag{A.4}$$

Eqs. (A.3)-(A.4) can then be replaced in Proposition 1:

$$\mu = \frac{\mu_T \sigma_M^2}{\sigma_T^2} \left(\widetilde{\beta} + \frac{\sigma_{M\Delta}}{\sigma_M^2} \widetilde{\beta}_\Delta \right) + \frac{\mu_T \sigma_\Delta^2}{\sigma_T^2} \left(\widetilde{\beta}_\Delta + \frac{\sigma_{M\Delta}}{\sigma_\Delta^2} \widetilde{\beta} \right). \tag{A.5}$$

This relation can be further simplified by eliminating $\sigma_{M\Delta}$ and μ_T. In order to do this, multiply Eq. (2) with M' and with Δ' to obtain two equations with two unknowns:

$$\mu_M = \frac{\mu_T \sigma_M^2}{\sigma_T^2} + \frac{\mu_T}{\sigma_T^2} \sigma_{M\Delta} \tag{A.6}$$

$$\mu_\Delta = \frac{\mu_T}{\sigma_T^2} \sigma_{M\Delta} + \frac{\mu_T \sigma_\Delta^2}{\sigma_T^2}, \tag{A.7}$$

which yields

$$\sigma_{M\Delta} = \frac{\mu_\Delta \sigma_M^2 - \mu_M \sigma_\Delta^2}{\mu_M - \mu_\Delta} \tag{A.8}$$

$$\mu_T = \frac{(\mu_M - \mu_\Delta) \sigma_T^2}{\sigma_M^2 - \sigma_\Delta^2}. \tag{A.9}$$

Replacing $\sigma_{M\Delta}$ and μ_T from above in Eq. (A.5) yields

$$\mu = \mu_M \widetilde{\beta} + \mu_\Delta \widetilde{\beta}_\Delta. \tag{A.10}$$

which is Eq. (5) of Corollary 1.1. Furthermore, solving for $\widetilde{\beta}$ and $\widetilde{\beta}_\Delta$ in (A.3)-(A.4) yields Eq. (6). This completes the proof of Corollary 1.1.

A.2 Proof of Proposition 2

The minimum-variance smart beta portfolio with expected return μ_{SB} is the solution w to

$$\min_{w} w'\Sigma w, \tag{A.11}$$

subject to

$$\mathbf{1}'_N w = 1, \quad \mu' w = \mu_{SB}, \quad M'\Sigma w = \sigma^2_M. \tag{A.12}$$

The last constraint imposes a constant covariance of σ^2_M with the market portfolio (or a beta of one for the smart beta fund). To solve this problem, set up the Lagrangian

$$\mathcal{L} = \min_{w} w'\Sigma w + \zeta_1(1 - \mathbf{1}'_N w) + \zeta_2(\mu_{SB} - \mu' w) + \zeta_3(\sigma^2_M - M'\Sigma w). \tag{A.13}$$

The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial w} = \Sigma w - \zeta_1 \mathbf{1}_N - \zeta_2 \mu - \zeta_3 \Sigma M = 0 \tag{A.14}$$

$$\frac{\partial \mathcal{L}}{\partial \zeta_1} = 1 - \mathbf{1}'_N w = 0 \tag{A.15}$$

$$\frac{\partial \mathcal{L}}{\partial \zeta_2} = \mu_{SB} - \mu' w = 0 \tag{A.16}$$

$$\frac{\partial \mathcal{L}}{\partial \zeta_3} = \sigma^2_M - M'\Sigma w = 0. \tag{A.17}$$

Hence, all minimum-variance smart beta portfolios are of the form

$$SB \equiv w^* = \zeta_1 \Sigma^{-1} \mathbf{1}_N + \zeta_2 \Sigma^{-1} \mu + \zeta_3 M. \tag{A.18}$$

which is Eq. (8) in Proposition 2. In order to determine the constants ζ_1, ζ_2, and ζ_3, use the three constraints and require that they be satisfied. This yields

$$\zeta_1 A + \zeta_2 B + \zeta_3 = 1 \tag{A.19}$$

$$\zeta_1 B + \zeta_2 C + \zeta_3 \mu_M = \mu_{SB} \tag{A.20}$$

$$\zeta_1 + \zeta_2 \mu_M + \zeta_3 \sigma^2_M = \sigma^2_M, \tag{A.21}$$

where

$$A \equiv 1'_N \Sigma^{-1} 1'_N, \quad B \equiv \mathbf{1}'_N \Sigma^{-1} \mu, \quad C \equiv \mu' \Sigma^{-1} \mu, \tag{A.22}$$

and thus

$$\zeta_1 = \frac{(\mu_M - \mu_{SB})(\mu_M - B\sigma^2_M)}{C - 2B\mu_M + A\mu^2_M + B^2\sigma^2_M - AC\sigma^2_M} \tag{A.23}$$

$$\zeta_2 = \frac{(\mu_M - \mu_{SB})(A\sigma^2_M - 1)}{C - 2B\mu_M + A\mu^2_M + B^2\sigma^2_M - AC\sigma^2_M} \tag{A.24}$$

$$\zeta_3 = \frac{C + A\mu_M\mu_{SB} - B(\mu_M + \mu_{SB}) + B^2\sigma^2_M - AC\sigma^2_M}{C - 2B\mu_M + A\mu^2_M + B^2\sigma^2_M - AC\sigma^2_M}. \tag{A.25}$$

Multiplying Eq. (A.18) with $(SB)'\Sigma$ yields the variance of returns of portfolio SB:

$$\sigma_{SB}^2 = \zeta_1 + \zeta_2\mu_{SB} + \zeta_3\sigma_M^2. \tag{A.26}$$

This completes the proof of Proposition 2.

A.3 Proof of Proposition 3

We solve for a linear equilibrium of the economy in which market-to-book ratios satisfy

$$\frac{P}{K} = \alpha_0 F + \xi_0 M + \alpha\widetilde{F} + gG + \xi\widetilde{M}, \tag{A.27}$$

where α_0, α, and g are N-dimensional vectors and ξ_0 and ξ are $N \times N$ matrices, all of which will be determined in equilibrium by imposing the market clearing condition (14).

Each investor i forms expectations about excess returns based on her information set:

$$\mathcal{F}_i = \{V_i, G, P\}. \tag{A.28}$$

It will be convenient to isolate the informational part of market-to-book ratios by writing

$$\frac{P^a}{K} \equiv \frac{P}{K} - \alpha_0 F - \xi_0 M - gG = \alpha\widetilde{F} + \xi\widetilde{M}, \tag{A.29}$$

This equation shows that each market-to-book ratio is a noisy signal on $\widetilde{F}$. The precision of each one of these signals is endogenously determined in equilibrium.

We use the *Projection Theorem* (see, e.g., DeGroot, 2005), which we restate here for convenience.

Projection Theorem. *Consider the n-dimensional normal random variable*

$$(\theta, s) \sim \mathcal{N}\left(\begin{bmatrix} \mu_\theta \\ \mu_s \end{bmatrix}, \begin{bmatrix} \Sigma_{\theta,\theta} & \Sigma_{\theta,s} \\ \Sigma_{s,\theta} & \Sigma_{s,s} \end{bmatrix}\right). \tag{A.30}$$

Provided $\Sigma_{s,s}$ is non-singular, the conditional density of θ given s is normal with conditional mean and conditional variance-covariance matrix:

$$\mathbb{E}[\theta|s] = \mu_\theta + \Sigma_{\theta,s}\Sigma_{s,s}^{-1}(s - \mu_s) \tag{A.31}$$

$$\mathrm{Var}[\theta|s] = \Sigma_{\theta,\theta} - \Sigma_{\theta,s}\Sigma_{s,s}^{-1}\Sigma_{s,\theta}. \tag{A.32}$$

Stack all the information of investor i, both private and public, into a single vector

$$S_i = \begin{bmatrix} P^a/K \\ V_i \\ G \end{bmatrix} = \begin{bmatrix} \alpha \\ 1 \\ 1 \end{bmatrix}\widetilde{F} + \begin{bmatrix} \xi & \mathbf{0}_{N\times 1} & \mathbf{0}_{N\times 1} \\ \mathbf{0}_{1\times N} & 1 & 0 \\ \mathbf{0}_{1\times N} & 0 & 1 \end{bmatrix}\begin{bmatrix} \widetilde{M} \\ v_i \\ v \end{bmatrix} \equiv H\widetilde{F} + \Theta\begin{bmatrix} \widetilde{M} \\ v_i \\ v \end{bmatrix}, \tag{A.33}$$

where the vector of noise in the signals, $[\widetilde{M}\ \ v_i\ \ v]'$, is jointly Gaussian with covariance matrix:

$$\Sigma = \begin{bmatrix} \tau_M^{-1}\,\mathbb{I}_N & \mathbf{0}_{N\times 1} & \mathbf{0}_{N\times 1} \\ \mathbf{0}_{1\times N} & \tau_v^{-1} & 0 \\ \mathbf{0}_{1\times N} & 0 & \tau_G^{-1} \end{bmatrix}. \tag{A.34}$$

We define

$$r \equiv (\Theta\Sigma\Theta')^{-1} = \begin{bmatrix} \tau_M(\xi\xi')^{-1} & \mathbf{0}_{N\times 1} & \mathbf{0}_{N\times 1} \\ \mathbf{0}_{1\times N} & \tau_v & 0 \\ \mathbf{0}_{1\times N} & 0 & \tau_G \end{bmatrix}, \tag{A.35}$$

and obtain that an investor i's total precision on the common factor satisfies

$$\tau \equiv \mathrm{Var}[\widetilde{F}|\mathcal{F}_i]^{-1} = \tau_F + H'rH = \tau_F + \tau_G + \tau_v + \tau_M\alpha'(\xi\xi')^{-1}\alpha. \tag{A.36}$$

To obtain (A.36), first replace $\Sigma_{\theta,\theta} = 1/\tau_F$, $\Sigma_{\theta,s} = H'/\tau_F$, $\Sigma_{s,\theta} = H/\tau_F$, and $\Sigma_{s,s} = HH'/\tau_F + \Theta\Sigma\Theta'$ in Eq. (A.32), then use the Woodbury matrix identity.

The precision τ is the same across investors. Furthermore, investor i's expectation of $\widetilde{F}$ satisfies:

$$\mathbb{E}[\widetilde{F}|\mathcal{F}_i] = \frac{1}{\tau}H'rS_i = \frac{1}{\tau}\begin{bmatrix}\tau_M\alpha'(\xi\xi')^{-1} & \tau_v & \tau_G\end{bmatrix} S_i. \tag{A.37}$$

To obtain (A.37), start from (A.31):

$$\mathbb{E}[\widetilde{F}|\mathcal{F}_i] = \frac{H'}{\tau_F}\left(\frac{HH'}{\tau_F} + \Theta\Sigma\Theta'\right)^{-1} S_i \tag{A.38}$$

$$= \frac{1}{\tau_F}H'r - \frac{1}{\tau\tau_F}H'rHH'r, \tag{A.39}$$

and replace $H'rH = \tau - \tau_F$ in the last term on the right hand side. Replacing S_i in (A.37) yields

$$\mathbb{E}[\widetilde{F}|\mathcal{F}_i] = \frac{1}{\tau}\left[(\tau - \tau_F - \tau_G)\widetilde{F} + \tau_G G + \tau_M\alpha'(\xi\xi')^{-1}\xi\widetilde{M} + \tau_v v_i\right], \tag{A.40}$$

where we have used the definition of the total precision (A.36) for the term that multiplies $\widetilde{F}$. It follows that average market expectation of future payoffs is

$$\widehat{\mathbb{E}}[D] = \int_i \mathbb{E}[D|\mathcal{F}_i]di = K\Phi F + K\Phi\frac{1}{\tau}\left[(\tau - \tau_F - \tau_G)\widetilde{F} + \tau_G G + \tau_M\alpha'(\xi\xi')^{-1}\xi\widetilde{M}\right], \tag{A.41}$$

and the covariance matrix of future payoffs is

$$\widehat{\Sigma} \equiv \mathrm{Var}[D|\mathcal{F}_i] = K^2\left(\frac{1}{\tau}\Phi\Phi' + \frac{1}{\tau_\epsilon}\mathbb{I}_N\right). \tag{A.42}$$

The market-clearing condition (14) implies

$$P = \widehat{\mathbb{E}}[D] - \gamma\widehat{\Sigma}(M + \widetilde{M}). \tag{A.43}$$

Thus

$$\frac{P}{K} = \Phi F + \Phi\frac{1}{\tau}\left[(\tau - \tau_F - \tau_G)\widetilde{F} + \tau_G G + \tau_M(\xi^{-1}\alpha)'\widetilde{M}\right] - \gamma\frac{\widehat{\Sigma}}{K}(M + \widetilde{M}) \tag{A.44}$$

where we have used the simplification $\alpha'(\xi\xi')^{-1}\xi = (\xi^{-1}\alpha)'$. This yields

$$\alpha_0 = \Phi, \quad \xi_0 = -\gamma\frac{\widehat{\Sigma}}{K}, \quad \alpha = \Phi\frac{\tau - \tau_F - \tau_G}{\tau}, \quad g = \Phi\frac{\tau_G}{\tau}, \tag{A.45}$$

and

$$\xi = \Phi\frac{\tau_M}{\tau}(\xi^{-1}\alpha)' - \gamma K\left(\frac{1}{\tau}\Phi\Phi' + \frac{1}{\tau_\epsilon}\mathbb{I}_N\right). \tag{A.46}$$

Multiply both sides of Eq. (A.46) by $\xi^{-1}\alpha$ (to the right):

$$\alpha = \Phi\frac{\tau_M}{\tau}(\xi^{-1}\alpha)'\xi^{-1}\alpha - \gamma K\left(\frac{1}{\tau}\Phi\Phi' + \frac{1}{\tau_\epsilon}\mathbb{I}_N\right)\xi^{-1}\alpha, \tag{A.47}$$

and then recognize that $\tau_M(\xi^{-1}\alpha)'\xi^{-1}\alpha = \tau_M\alpha'(\xi\xi')^{-1}\alpha = \tau - \tau_F - \tau_G - \tau_v$ (from Eq. A.36), which can be replaced above, together with the solution for α to obtain (after multiplication with τ):

$$\Phi\tau_v = -\gamma K\left(\Phi\Phi' + \frac{\tau}{\tau_\epsilon}\mathbb{I}_N\right)\xi^{-1}\alpha, \tag{A.48}$$

which leads to an equation for $\xi^{-1}\alpha$:

$$\xi^{-1}\alpha = -\frac{\tau_v}{\gamma K}\left(\Phi\Phi' + \frac{\tau}{\tau_\epsilon}\mathbb{I}_N\right)^{-1}\Phi. \tag{A.49}$$

Multiply both sides with Φ' (to the left):

$$\Phi'\xi^{-1}\alpha = -\frac{\tau_v}{\gamma K}\Phi'\left(\Phi\Phi' + \frac{\tau}{\tau_\epsilon}\mathbb{I}_N\right)^{-1}\Phi = -\frac{\tau_v\tau_\epsilon\Phi'\Phi}{\gamma K(\tau + \tau_\epsilon\Phi'\Phi)}, \tag{A.50}$$

where the second equality follows from the Woodbury matrix identity. Conjecture

$$\xi^{-1}\alpha \equiv -\frac{\sqrt{\tau_P}}{\sqrt{\tau_M}}\Phi, \tag{A.51}$$

where τ_P is an unknown positive scalar. Replacing Eq. (A.51) in Eq. (A.36) yields the total precision τ as a function of this scalar:

$$\tau = \tau_F + \tau_G + \tau_v + \tau_P\Phi'\Phi. \tag{A.52}$$

Furthermore, replacing the conjecture (A.51) in Eq. (A.50) yields

$$\frac{\sqrt{\tau_P}}{\sqrt{\tau_M}} = \frac{\tau_v\tau_\epsilon}{\gamma K(\tau + \tau_\epsilon\Phi'\Phi)} \tag{A.53}$$

which leads to a cubic equation in τ_P:

$$\tau_P\left[\tau_F + \tau_v + \tau_G + (\tau_P + \tau_\epsilon)\Phi'\Phi\right]^2 = \frac{\tau_M\tau_\epsilon^2\tau_v^2}{\gamma^2K^2}. \tag{A.54}$$

The discriminant of this equation is strictly negative and thus it has a unique real root. Since it cannot have a negative root (the right hand side is strictly positive), it follows that τ_P is a unique positive scalar. The conjecture (A.51) can now be replaced in (A.46) to obtain the undetermined coefficients ξ:

$$\xi = -\frac{\gamma K + \sqrt{\tau_M \tau_P}}{\tau} \Phi\Phi' - \frac{\gamma K}{\tau_\epsilon} \mathbb{I}_N \,. \tag{A.55}$$

This completes the proof of Proposition 3.

A.4 Proof of Corollary 3.3

For this proof we will make the following assumptions:

Assumption 1. *There is no ex-ante proportionality relation between the unconditional market portfolio M and the vector of firms' loadings on the common productivity factor Φ.*

Assumption 1 ensures that we keep the setup as general as possible, excluding pathological cases with an exogenous *perfect* relationship between firms' market capitalizations and their exposure to the common productivity factor.

Assumption 2. $M'\Phi > 0$.

Assumption 2 eliminates the uninteresting case $M'\Phi = 0$ (zero market exposure to the common factor), and is without loss of generality (if $M'\Phi < 0$, one can simply switch the sign of the common factor).

Setting $x = \Sigma^{1/2}M$ and $y = \Sigma^{-1/2}\mu$, we have $\sigma_M = \|x\|$ and $\sqrt{\mu'\Sigma^{-1}\mu} = \|y\|$, where $\|\cdot\|$ denotes the norm. The *Cauchy-Schwartz* inequality states that

$$\|x\|\|y\| \geq x'y = M'\Sigma^{1/2}\Sigma^{-1/2}\mu = \mu_M. \tag{A.56}$$

where we have used the properties of symmetric positive-definite matrices for Σ. Thus,

$$\frac{\mu_M}{\sigma_M} \leq \sqrt{\mu'\Sigma^{-1}\mu}. \tag{A.57}$$

The relation (A.57) holds with equality if and only if x is proportional to y, or

$$\mu \propto \Sigma M. \tag{A.58}$$

To show that the proportionality relation (A.58) is cannot hold for the econometrician, we first compute Σ by using the law of total variance:

$$\begin{aligned}\Sigma &= \widehat{\Sigma} + \mathrm{Var}[\mathbb{E}[D - P|\mathcal{F}_i]] &\text{(A.59)}\\ &= \widehat{\Sigma} + \mathrm{Var}\left[\widehat{\mathbb{E}}[D] + K\Phi\frac{\tau_v}{\tau}v_i\right] &\text{(A.60)}\\ &= \widehat{\Sigma} + \frac{\gamma^2}{\tau_M}\widehat{\Sigma}^2 + \frac{K^2\tau_v}{\tau^2}\Phi\Phi' &\text{(A.61)}\end{aligned}$$

Replace (A.42) above to obtain

$$\Sigma = \left(\frac{K^2(\tau+\tau_v)}{\tau^2} + \frac{K^4\gamma^2\Phi'\Phi}{\tau^2\tau_M} + \frac{2K^4\gamma^2}{\tau\tau_M\tau_\epsilon}\right)\Phi\Phi' + \left(\frac{K^2}{\tau_\epsilon} + \frac{K^4\gamma^2}{\tau_M\tau_\epsilon^2}\right)\mathbb{I}_N \,. \tag{A.62}$$

Replace $\mathbb{I}_N$ from (A.42) or $\Phi\Phi'$ from (A.42) to write Σ in two equivalent forms:

$$\Sigma = c_1\widehat{\Sigma} + c_2\Phi\Phi' \tag{A.63}$$

$$\Sigma = c_3\widehat{\Sigma} - c_4\,\mathbb{I}_N \tag{A.64}$$

where c_1, c_2, c_3, and c_4 are positive scalars:

$$c_1 = 1 + \frac{K^2\gamma^2}{\tau_M\tau_\epsilon} > 0, \quad c_2 = \frac{K^2\tau_v}{\tau^2} + \frac{\gamma^2K^4(\tau + \tau_\epsilon\Phi'\Phi)}{\tau^2\tau_M\tau_\epsilon} > 0, \tag{A.65}$$

and

$$c_3 = 1 + \frac{\tau_v}{\tau} + \frac{K^2\gamma^2(2\tau + \tau_\epsilon\Phi'\Phi)}{\tau\tau_M\tau_\epsilon} > 0, \quad c_4 = \frac{K^2\tau_v}{\tau\tau_\epsilon} + \frac{\gamma^2K^4(\tau + \tau_\epsilon\Phi'\Phi)}{\tau\tau_M\tau_\epsilon^2} > 0. \tag{A.66}$$

Multiply Equations (A.63)-(A.64) with M:

$$\Sigma M = c_1\widehat{\Sigma}M + c_2(\Phi'M)\Phi \tag{A.67}$$

$$\Sigma M = c_3\widehat{\Sigma}M - c_4M. \tag{A.68}$$

Since $\widehat{\Sigma}M$ and μ are proportional (Corollary 3.1), it follows that (A.58) implies $\mu \propto \Phi$ and $\mu \propto M$. This implies $M \propto \Phi$, contradicting Assumption 1, which completes the proof of Corollary 3.3.

A.5 Proof of Proposition 4

Start from the true CAPM relation (Corollary 3.1):

$$\mu = \frac{\mu_M}{\widehat{\sigma}_M^2}\widehat{\Sigma}M, \tag{A.69}$$

and replace $\widehat{\Sigma}$ from (A.63):

$$\mu = \frac{\mu_M\sigma_M^2}{c_1\widehat{\sigma}_M^2}\beta - \frac{\mu_M c_2 M'\Phi}{c_1\widehat{\sigma}_M^2}\Phi. \tag{A.70}$$

We further know that average market-to-book ratios are

$$\frac{\mathbb{E}[P]}{K} = \Phi F - \frac{\mu}{K}, \tag{A.71}$$

from which we can replace Φ in (A.70) and solve for μ. This yields

$$\mu = \lambda_1\beta + \lambda_2\frac{\mathbb{E}[P]}{K}, \tag{A.72}$$

with

$$\lambda_1 = \frac{FK\mu_M\sigma_M^2}{c_1FK\widehat{\sigma}_M^2 + c_2\mu_M M'\Phi} \quad \text{and} \quad \lambda_2 = -\frac{c_2K\mu_M M'\Phi}{c_1FK\widehat{\sigma}_M^2 + c_2\mu_M M'\Phi}. \tag{A.73}$$

Assumption 2 ensures that $\lambda_1 > 0$ and $\lambda_2 < 0$. This completes the proof of Proposition 4.

A.6 Proof of Corollary 4.1

Replacing Φ from (24) in (A.70) yields (25), with

$$\eta_0 = -\frac{c_2 M' \Phi \mu_M}{c_1 F \widehat{\sigma}_M^2}, \quad \eta_1 = \frac{\mu_M \sigma_M^2}{c_1 \widehat{\sigma}_M^2}, \quad \text{and} \quad \eta_2 = -\frac{a c_2 M' \Phi \mu_M}{c_1 \widehat{\sigma}_M^2}. \tag{A.74}$$

This completes the proof of Corollary 4.1.

A.7 Proof of Proposition 5

This appendix solves an extension of the model when assets' payoffs are driven by multiple factors. We start by conjecturing that prices satisfy

$$P = \underbrace{\alpha_0}_{N\times K\times} F + \underbrace{\xi_0}_{N\times N} M + \underbrace{\alpha}_{N\times K} \mathbf{F} + \underbrace{g}_{N\times K} \mathbf{G} + \underbrace{\beta}_{N\times N} \widetilde{M}. \tag{A.75}$$

Since agents observe $\mathbf{G}$, M, and F the effective price signal is

$$P^a \equiv P - g\mathbf{G} - \alpha_0 F - \xi_0 M = \alpha \mathbf{F} + \beta M. \tag{A.76}$$

Regrouping all signals in a vector we obtain

$$S^i = \begin{pmatrix} P^a \\ \mathbf{V}^i \\ \mathbf{G} \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha \\ \mathbf{I}_K \\ \mathbf{I}_K \end{pmatrix}}_{H \ (N+2K)\times K} \mathbf{F} + \underbrace{\begin{pmatrix} \beta & \mathbf{0}_{N\times K} & \mathbf{0}_{N\times K} \\ \mathbf{0}_{K\times N} & \mathbf{I}_K & \mathbf{0}_{K\times K} \\ \mathbf{0}_{K\times N} & \mathbf{0}_{K\times K} & \mathbf{I}_K \end{pmatrix}}_{\Theta \ (N+2K)\times(N+2K)} \begin{pmatrix} M \\ v^i \\ v \end{pmatrix} \tag{A.77}$$

with

$$\begin{pmatrix} M \\ v^i \\ v \end{pmatrix} \sim \mathscr{N}\left(\mathbf{0}, \underbrace{\begin{pmatrix} \tau_M^{-1}\mathbf{I}_N & \mathbf{0}_{N\times K} & \mathbf{0}_{N\times K} \\ \mathbf{0}_{K\times N} & \tau_v^{-1}\mathbf{I}_K & \mathbf{0}_{K\times K} \\ \mathbf{0}_{K\times N} & \mathbf{0}_{K\times K} & \tau_G^{-1}\mathbf{I}_K \end{pmatrix}}_{\Sigma \ (N+2K)\times(N+2K)} \right). \tag{A.78}$$

Using these matrices we now define a $(N + 2K) \times (N + 2K)-$matrix:

$$R \equiv (\Theta\Sigma\Theta')^{-1} = \begin{pmatrix} (\beta\beta')^{-1}\tau_M & \mathbf{0}_{N\times K} & \mathbf{0}_{N\times K} \\ \mathbf{0}_{K\times N} & \mathbf{I}_K \tau_v & \mathbf{0}_{K\times K} \\ \mathbf{0}_{K\times N} & \mathbf{0}_{K\times K} & \mathbf{I}_K \tau_G \end{pmatrix}. \tag{A.79}$$

The projection theorem then implies that

$$\tau \equiv \mathbb{V}[\mathbf{F}|\mathscr{F}_i]^{-1} = \mathbf{I}_K \tau_F + H'RH \tag{A.80}$$

$$= (\tau_F + \tau_v + \tau_G)\mathbf{I}_K + \alpha'(\beta\beta')^{-1}\alpha\tau_M \tag{A.81}$$

The projection theorem also yields

$$\mathbb{E}_i[\mathbf{F}|\mathscr{F}_i] = \tau^{-1}H'RS_i \tag{A.82}$$

$$= \tau^{-1}\left(\tau_M\alpha'(\beta\beta')^{-1}(\alpha\mathbf{F} + \beta\widetilde{M}) + \tau_v(\mathbf{F} + v_i) + \tau_G(\mathbf{F} + v)\right) \tag{A.83}$$

$$= \tau^{-1}\left((\tau - \tau_F\mathbf{I}_K - \tau_G\mathbf{I}_K)\mathbf{F} + \tau_M\alpha'(\beta\beta')^{-1}\beta\widetilde{M} + \tau_v v_i + \tau_G\mathbf{G}\right). \tag{A.84}$$

It follows that average expectations of future payoffs satisfy

$$\widehat{\mathbb{E}}[D] \equiv \int_i \mathbb{E}_i[D|\mathscr{F}_i]\mathrm{d}i = \Phi F + \Phi\tau^{-1}\left((\tau - \tau_F\mathbf{I}_K - \tau_G\mathbf{I}_K)\mathbf{F} + \tau_M\alpha'(\beta\beta')^{-1}\beta\widetilde{M} + \tau_G\mathbf{G}\right). \tag{A.85}$$

and the covariance matrix of future payoffs satisfies:

$$\widehat{\Sigma} \equiv \mathbb{V}[D|\mathscr{F}_i] = \Phi\tau^{-1}\Phi' + \tau_\epsilon^{-1}\mathbf{I}_N. \tag{A.86}$$

The market-clearing condition then requires that

$$P = \widehat{\mathbb{E}}[D] - \gamma\widehat{\Sigma}(M + \widetilde{M}) \tag{A.87}$$

$$= \Phi F + \Phi\tau^{-1}\left((\tau - \tau_F\mathbf{I}_K - \tau_G\mathbf{I}_K)\mathbf{F} + \tau_M(\beta^{-1}\alpha)'\widetilde{M} + \tau_G\mathbf{G}\right) - \gamma\widehat{\Sigma}(M + \widetilde{M}). \tag{A.88}$$

Separating variables we obtain the following system of equations:

$$\alpha_0 = \Phi, \quad \xi_0 = -\gamma\widehat{\Sigma}, \quad g = \tau_G\Phi\tau^{-1}, \quad \alpha = \Phi\tau^{-1}(\tau - \tau_F\mathbf{I}_K - \tau_G\mathbf{I}_K), \tag{A.89}$$

and

$$\beta = \tau_M\Phi\tau^{-1}(\beta^{-1}\alpha)' - \gamma\left(\Phi\tau^{-1}\Phi' + \tau_\epsilon^{-1}\mathbf{I}_N\right) \tag{A.90}$$

We can reduce the size of this system of equations. Post-multiply both sides of the above equation by $\beta^{-1}\alpha$:

$$\alpha = \tau_M\Phi\tau^{-1}(\beta^{-1}\alpha)'\beta^{-1}\alpha - \gamma\left(\Phi\tau^{-1}\Phi' + \tau_\epsilon^{-1}\mathbf{I}_N\right)\beta^{-1}\alpha \tag{A.91}$$

Observing that

$$\tau_M\Phi\tau^{-1}(\beta^{-1}\alpha)'\beta^{-1}\alpha = \Phi\tau^{-1}\left(\tau - (\tau_F + \tau_G + \tau_v)\mathbf{I}_K\right) \equiv \alpha - \tau_v\Phi\tau^{-1}, \tag{A.92}$$

we obtain

$$\tau_v\Phi\tau^{-1} = -\gamma\left(\Phi\tau^{-1}\Phi' + \tau_\epsilon^{-1}\mathbf{I}_N\right)\beta^{-1}\alpha, \tag{A.93}$$

which yields an equation for the vector of signal-to-noise ratio:

$$\beta^{-1}\alpha = -\frac{\tau_v}{\gamma}\left(\Phi\tau^{-1}\Phi' + \tau_\epsilon^{-1}\mathbf{I}_N\right)^{-1}\Phi\tau^{-1}. \tag{A.94}$$

Pre-multiply this equation by $\tau^{-1}\Phi'$ and use Woodbury matrix identity that implies:

$$\tau^{-1}\Phi'\left(\Phi\tau^{-1}\Phi' + \tau_\epsilon^{-1}\mathbf{I}_N\right)^{-1}\Phi\tau^{-1} = \tau^{-1} - (\tau + \tau_\epsilon\Phi'\Phi)^{-1} \tag{A.95}$$

to conclude that

$$\tau^{-1}\Phi'\beta^{-1}\alpha = -\frac{\tau_v}{\gamma}\left(\tau^{-1} - (\tau + \tau_\epsilon \Phi'\Phi)^{-1}\right). \tag{A.96}$$

Conjecture that

$$\beta^{-1}\alpha \equiv -\frac{1}{\sqrt{\tau_M}}\Phi\tau_P, \tag{A.97}$$

where τ_P is a $K \times K-$symmetric matrix of $K(K+1)/2$ unknown coefficients. Replacing this conjecture in the expression for total precision in Eq. (A.80):

$$\tau \equiv (\tau_F + \tau_v + \tau_G)\mathbf{I}_K + \tau_P\Phi'\Phi\tau_P. \tag{A.98}$$

Further replacing the conjecture in Eq. (A.96) produces a matrix equation for τ_P:

$$\tau^{-1}\Phi'\Phi\tau_P = \sqrt{\tau_M}\frac{\tau_v}{\gamma}\left(\tau^{-1} - (\tau + \tau_\epsilon\Phi'\Phi)^{-1}\right), \tag{A.99}$$

which, premultiplying by τ, can be rewritten as

$$\Phi'\Phi\tau_P = \sqrt{\tau_M}\frac{\tau_v}{\gamma}\left(\mathbf{I}_K - (\mathbf{I}_K + \tau_\epsilon\tau^{-1}\Phi'\Phi)^{-1}\right). \tag{A.100}$$

Once we have a solution for τ_P we can subsitute the conjecture we obtain the matrix β as:

$$\beta = -\sqrt{\tau_M}\Phi\tau^{-1}\tau_P'\Phi' - \gamma\left(\Phi\tau^{-1}\Phi' + \frac{1}{\tau_\epsilon}\mathbf{I}_N\right). \tag{A.101}$$

A.8 Proof of Proposition 6

As in the single factor case, the market-clearing condition implies

$$\widehat{\mathbb{E}}[R] = \gamma\widehat{\Sigma}(M + \widetilde{M}), \tag{A.102}$$

a relation that can be represented in the traditional CAPM form. To construct this relation as measured by the econometrician, we use the law of total covariance:

$$\Sigma = \widehat{\Sigma} + \mathbb{V}\left[\widehat{\mathbb{E}}[R] + \tau_v\Phi\tau^{-1}v_i\right] \tag{A.103}$$

$$= \widehat{\Sigma} + \frac{\gamma^2}{\tau_M}\widehat{\Sigma}^2 + \tau_v\Phi\tau^{-1}\tau^{-1}\Phi', \tag{A.104}$$

where the second line follows from substituting the market-clearing condition above. Furthermore, the conditional covariance matrix of returns satisfies:

$$\widehat{\Sigma} = \Phi\tau^{-1}\Phi' + \tau_\epsilon^{-1}\mathbf{I}. \tag{A.105}$$

Substituting one relation into the other we obtain

$$\Sigma = \Phi\tau^{-1}\Phi' + \tau_\epsilon^{-1}\mathbf{I} + \frac{\gamma^2}{\tau_M}(\Phi\tau^{-1}\Phi' + \tau_\epsilon^{-1}\mathbf{I})(\Phi\tau^{-1}\Phi' + \tau_\epsilon^{-1}\mathbf{I}) + \tau_v\Phi\tau^{-1}\tau^{-1}\Phi' \tag{A.106}$$

$$= \left(1 + \frac{\gamma^2}{\tau_M\tau_\epsilon}\right)\widehat{\Sigma} + \frac{\gamma^2}{\tau_M\tau_\epsilon}\Phi\tau^{-1}\Phi' + \Phi\tau^{-1}\left(\frac{\gamma^2}{\tau_M}\Phi'\Phi + \tau_v\mathbf{I}_K\right)\tau^{-1}\Phi' \tag{A.107}$$

Let us further write:

$$\tau^{-1} = \begin{pmatrix} \omega_{11} & \omega_{12} & \dots & \omega_{1K} \\ \omega_{12} & \omega_{22} & \dots & \omega_{2K} \\ \vdots & \vdots & \vdots & \vdots \\ \omega_{1K} & \omega_{2K} & \dots & \omega_{KK} \end{pmatrix}, \tag{A.108}$$

and

$$\Phi \equiv \begin{pmatrix} \Phi_1 & \Phi_2 & \dots & \Phi_K \end{pmatrix}. \tag{A.109}$$

We can then write:

$$\Phi\tau^{-1}\Phi' = \sum_{j=1}^{K}\sum_{k=1}^{K}\omega_{kj}\Phi_k\Phi_j', \tag{A.110}$$

$$\Phi\tau^{-1}\tau^{-1}\Phi' = \sum_{j=1}^{K}\sum_{k=1}^{K}\left(\sum_{n=1}^{K}\omega_{kn}\omega_{nj}\right)\Phi_k\Phi_j'. \tag{A.111}$$

We further need to compute:

$$\Phi\tau^{-1}\Phi'\Phi\tau^{-1}\Phi' = \sum_{n=1}^{K}\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{k=1}^{K}\omega_{in}\omega_{kj}\Phi_i\Phi_n'\Phi_k\Phi_j'. \tag{A.112}$$

We first note that:

$$\Phi_i\Phi_n'\Phi_k\Phi_j' = (\Phi_n'\Phi_k)\Phi_i\Phi_j'. \tag{A.113}$$

Substituting back we get:

$$\Phi\tau^{-1}\Phi'\Phi\tau^{-1}\Phi' = \sum_{n=1}^{K}\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{k=1}^{K}\omega_{in}\omega_{kj}(\Phi_n'\Phi_k)\Phi_i\Phi_j' \tag{A.114}$$

$$= \sum_{j=1}^{K}\sum_{i=1}^{K}\left(\sum_{k=1}^{K}\sum_{n=1}^{K}\omega_{in}\omega_{kj}(\Phi_n'\Phi_k)\right)\Phi_i\Phi_j' \tag{A.115}$$

For convenience, we relabel the indices to obtain:

$$\Phi\tau^{-1}\Phi'\Phi\tau^{-1}\Phi' = \sum_{j=1}^{K}\sum_{k=1}^{K}\left(\sum_{i=1}^{K}\sum_{n=1}^{K}\omega_{kn}\omega_{ij}(\Phi_n'\Phi_i)\right)\Phi_k\Phi_j' \tag{A.116}$$

We can then compute:

$$\Phi\tau^{-1}\left(\frac{\gamma^2}{\tau_M}\Phi'\Phi+\tau_v\mathbf{I}_K\right)\tau^{-1}\Phi'+\frac{\gamma^2}{\tau_M\tau_\epsilon}\Phi\tau^{-1}\Phi' \tag{A.117}$$

$$=\sum_{j=1}^K\sum_{k=1}^K\left(\frac{\gamma^2}{\tau_M\tau_\epsilon}\omega_{kj}+\tau_v\sum_{n=1}^K\omega_{nj}\omega_{kn}+\frac{\gamma^2}{\tau_M}\sum_{i=1}^K\sum_{n=1}^K\omega_{kn}\omega_{ij}(\Phi_n'\Phi_i)\right)\Phi_k\Phi_j' \tag{A.118}$$

$$=\sum_{j=1}^K\sum_{k=1}^K\underbrace{\left(\frac{\gamma^2}{\tau_M\tau_\epsilon}\omega_{kj}+\sum_{n=1}^K\omega_{kn}\left(\tau_v\omega_{nj}+\frac{\gamma^2}{\tau_M}\sum_{i=1}^K\omega_{ij}\Phi_n'\Phi_i\right)\right)}_{\equiv c_{kj}}\Phi_k\Phi_j', \tag{A.119}$$

and thus

$$\Sigma=\left(1+\frac{\gamma^2}{\tau_M\tau_\epsilon}\right)\widehat{\Sigma}+\sum_{j=1}^K\sum_{k=1}^K c_{kj}\Phi_k\Phi_j'. \tag{A.120}$$

We can write expected returns as

$$\mu=\frac{\mu_M}{\widehat{\sigma}_M^2}\left(1+\frac{\gamma^2}{\tau_M\tau_\epsilon}\right)^{-1}\left(\sigma_M^2\beta-\sum_{k=1}^K\left(\sum_{j=1}^K\phi_j c_{kj}\right)\Phi_k\right), \tag{A.121}$$

where $\phi_j\equiv\Phi_j'M$ denotes the average loading on factor j. Similarly, we can obtain a relation between β_Δ and the K factors:

$$\beta_\Delta=\frac{\sigma_T^2\sigma_\Delta^2}{\mu_T}\left(\sigma_M^2\left(\frac{\mu_M}{\widehat{\sigma}_M^2}\left(1+\frac{\gamma^2}{\tau_M\tau_\epsilon}\right)^{-1}-\frac{\mu_T}{\sigma_T^2}\right)\beta-\frac{\mu_M}{\widehat{\sigma}_M^2}\left(1+\frac{\gamma^2}{\tau_M\tau_\epsilon}\right)^{-1}\sum_{j=1}^K\left(\phi_j\sum_{k=1}^K c_{kj}\right)\Phi_k\right) \tag{A.122}$$

$$=\frac{\sigma_T^2\sigma_\Delta^2}{\mu_T}\left(\sigma_M^2\left(\frac{\gamma\tau_M\tau_\epsilon}{\tau_M\tau_\epsilon+\gamma^2}-B\right)\beta-\frac{\mu_M}{\widehat{\sigma}_M^2}\left(1+\frac{\gamma^2}{\tau_M\tau_\epsilon}\right)^{-1}\sum_{j=1}^K\left(\phi_j\sum_{k=1}^K c_{kj}\right)\Phi_k\right) \tag{A.123}$$

We can further rewrite the coefficient $B\equiv\mu'\Sigma^{-1}\mathbf{1}$ as follows:

$$B\equiv\mu'\Sigma^{-1}\mathbf{1}=\mathbf{1}'\Sigma^{-1}\mu \tag{A.124}$$

$$=\gamma\mathbf{1}'\Sigma^{-1}\widehat{\Sigma}M. \tag{A.125}$$

Now premultiply Eq. (A.120) by Σ^{-1} and rearrange to obtain:

$$\frac{\tau_M\tau_\epsilon}{\gamma^2+\tau_M\tau_\epsilon}\left(\mathbf{I}_N-\sum_{j=1}^K\sum_{k=1}^K c_{kj}\Sigma^{-1}\Phi_k\Phi_j'\right)=\Sigma^{-1}\widehat{\Sigma}, \tag{A.126}$$

which substituted back yields:

$$B = \frac{\gamma\tau_M\tau_\epsilon}{\gamma^2 + \tau_M\tau_\epsilon}\left(\underbrace{M'\mathbf{1}}_{\equiv 1} - \sum_{j=1}^{K}\sum_{k=1}^{K} c_{kj}\mathbf{1}'\Sigma^{-1}\Phi_k \underbrace{\Phi_j' M}_{\equiv \phi_j}\right) \tag{A.127}$$

$$= \frac{\gamma\tau_M\tau_\epsilon}{\gamma^2 + \tau_M\tau_\epsilon}\left(1 - \sum_{j=1}^{K}\phi_j\left(\sum_{k=1}^{K} c_{kj}\right)\mathbf{1}'\Sigma^{-1}\Phi_k\right) \tag{A.128}$$

Finally, substitute back in the relation for β_Δ and get:

$$\beta_\Delta = \frac{\sigma_T^2\sigma_\Delta^2}{\mu_T}\frac{\gamma\tau_M\tau_\epsilon}{\gamma^2 + \tau_M\tau_\epsilon}\sum_{j=1}^{K}\phi_j\sum_{k=1}^{K} c_{kj}\left(\underbrace{\mathbf{1}'\Sigma^{-1}\Phi_k}_{\equiv B_k}\sigma_M^2\beta - \Phi_k\right) \tag{A.129}$$

$$= \frac{\sigma_T^2\sigma_\Delta^2}{\mu_T}\frac{\gamma\tau_M\tau_\epsilon}{\gamma^2 + \tau_M\tau_\epsilon}\sum_{j=1}^{K}\phi_j\sum_{k=1}^{K} B_k c_{kj}\Sigma\left(M - \frac{\Sigma^{-1}\Phi_k}{B_k}\right) \tag{A.130}$$