Information Percolation, Momentum, and Reversal*

Daniel Andrei† Julien Cujean‡

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Abstract

The tendency of asset prices to trend over short horizons and slowly revert over long horizons—momentum and reversal—can find an explanation within a simple, rational model. The key ingredient is word-of-mouth communication, which we introduce in the standard noisy rational expectations framework. Word-of-mouth communication accelerates the information revelation and generates momentum in asset returns. Due to social interactions, investors with heterogeneous trading strategies optimally coexist in the marketplace—some agents are contrarians, others are momentum traders, yet momentum is not completely eliminated. Finally, word-of-mouth communication can propagate a rumor and a price run-up; the price reverts once the rumor subsides.

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†UCLA Anderson Graduate School of Management, 110 Westwood Plaza, Suite C420, Los Angeles, CA 90095, daniel.andrei@anderson.ucla.edu, www.danielandrei.net

‡Robert H. Smith School of Business, University of Maryland, 4466 Van Munching Hall, College Park, MD 20742, jcujean@rhsmith.umd.edu, www.juliencujean.com

1 Introduction

One of the most pervasive facts in finance is the "momentum" effect. Prices keep moving in the same direction over an horizon of 6 months to 1 year. Not only is this effect economically significant—around 1 % per month—but it also persists across countries and across asset classes, both in the cross section (Jegadeesh and Titman, 1993) and the time series (Moskowitz, Ooi, and Pedersen, 2012) of returns. Momentum is typically followed by a "reversal" effect (De Bondt and Thaler, 1985), whereby prices correct the past trend and move in the opposite direction over long-term horizons.

Short-term momentum and long-term reversal present a significant challenge to rational explanations—smart money investors should eliminate this predictable pattern. Behavioral theories therefore prevail: momentum traders, conservative investors, and attribution bias all generate momentum, while newswatchers, representativeness heuristic, and overconfidence all generate reversal.[1] But the wide prevalence of momentum and reversal, best highlighted in the recent study of Moskowitz et al. (2012), calls into question behavioral theories. First, the study documents momentum and reversal across different asset markets (e.g., currencies, bonds, equity indices, commodities); investors of different types, however, presumably trade in different markets. Second, the study fails to find a link between momentum and various measures of investor sentiment, suggesting that some of these theoretical models come short of identifying the sources of momentum and reversal.

This evidence calls for a *rational theory* of momentum and reversal, but the task of building such a theory seems formidable. If rational investors are able to consistently detect momentum and trade profitably on it, then why does momentum persist? This single question poses a major challenge to any rational theory.

In this paper we show that momentum and reversal can jointly find a simple, rational explanation. Surprisingly, our model departs minimally from a traditional "noisy rational-expectations equilibrium" model (Grossman and Stiglitz, 1980). We assume that information diffuses among a population of rational investors through interpersonal communication. Investors trade in centralized markets, but have to search for each other's private information. That is, in our model, markets are centralized but information is not.

Two learning channels arise in this novel kind of model: agents learn from the informational content of prices and through word-of-mouth communication. The former channel is standard in the literature; the latter channel represents our contribution. Most important, we show that *both* channels are necessary components for understanding how momentum and reversal emerge. This result is particularly important, for it suggests that the informational role of

[1] Hong and Stein (1999), Barberis, Shleifer, and Vishny (1998), Daniel, Hirshleifer, and Subrahmanyam (1998)

prices—commonly dismissed in leading theories—is actually a major driver of momentum and reversal.

Our model relies on a rational-expectations structure that we borrow from He and Wang (1995) and Brennan and Cao (1997). We consider an economy in which a large population of risk-averse investors trade a risky asset. Trading takes place at discrete dates until the asset's final payoff (the fundamental) is realized. All investors are initially endowed with a private signal about the fundamental. Investors then meet randomly at a fixed meeting intensity and exchange their information, exactly as in the information percolation model developed by Duffie, Malamud, and Manso (2009).

Depending on the intensity at which agents meet with each other, information percolation produces momentum in stock returns. More precisely, when the frequency of meetings is low, prices only slowly converge toward the fundamental. As a result, most investors engage in short-term speculation by accommodating noise traders' demand, causing stock returns to exhibit reversal. This reversal effect, commonly obtained in rational-expectations models (e.g., Wang 1993), may perhaps lead one to overlook the possibility of momentum. We show, however, that as information percolation intensifies, the information flow through prices is sufficiently strong for investors to engage in long-term speculation, causing stock returns to eventually exhibit momentum.

One fact about momentum is that it is easily identified from prices. Uninformed investors can therefore make profits by trading on momentum—buy when prices go up and sell when prices go down. Any theory of momentum must be consistent with this fact. We show that indeed, in our model, uninformed investors can identify momentum from prices and make profits on it by adopting this simple momentum strategy. Importantly, we show that such momentum traders do not eliminate momentum—momentum and momentum traders can coexist in the marketplace. A key element to generate this result is the heterogeneity in information endowments. While some traders find it optimal to trade on momentum, other traders find it optimal to be contrarians. Momentum therefore arises and persists in an economy with rational investors.

Not only does information percolation generate momentum, but it is also a natural propagator of rumors. If private information contains a rumor, this rumor will circulate from one person to another, creating a disconnect between the stock price and the fundamental. This phenomenon arises in our model if we assume that investors are aware of the *existence* of a rumor, but *not* of its magnitude. Then, as trading approaches its final round and the fundamental value is close to be revealed, the rumor subsides, producing a price reversal.

The rational-expectations literature focuses on the market as a whole and mostly ignores private interactions among investors; this is unfortunate, as prices being imperfect aggregators, social networks have an important role to play, in particular as an alternative channel conveying additional information through private discussions. For instance, Shiller (2000) writes: "*Word-of-mouth transmission of ideas appears to be an important contributor to day-to-day or hour-to-hour stock market fluctuations*," Stein (2008) describes conversations as being "*a central part of economic life*," and Shiller and Pound (1989) provide evidence that direct interpersonal communication is an important determinant of investors decisions. Two empirical contributions document the relevance of word-of-mouth communication for the serial correlation of stock returns. Hong, Lim, and Stein (2000) show that stocks with greater analyst coverage exhibit more momentum, providing evidence in favor of the underreaction hypothesis. Hong, Hong, and Ungureanu (2010) investigate, both theoretically and empirically, the relation between the serial correlation of returns and information diffusion based on word-of-mouth communication. Their conclusion is that a model of word-of-mouth communication is useful for understanding price momentum and dynamics around media events.[2]

2 Related Literature

Among the leading explanations for momentum and reversal, Daniel et al. (1998) and Hong and Stein (1999) are the ones most closely related to our paper.[3] While these two papers can be classified as behavioral explanations of momentum and reversal, they differ in one fundamental aspect. In Daniel et al. (1998) agents are overconfident (they overestimate the precision of private information) and have an attribution bias (their confidence increases if public information corroborates what they have already assumed). Overconfidence generates overreaction and reversals, whereas attribution bias generates continuing overreaction and momentum. Instead, Hong and Stein (1999) argue that the interaction between momentum traders and newswatchers generates underreaction followed by momentum and overreaction followed by reversal. The *continuing overreaction* explanation of Daniel et al. (1998) and the *underreaction* explanation of Hong and Stein (1999) seem to be incompatible; our model helps

[2] A comprehensive investigation of word-of-mouth communication among traders in financial markets is provided by Shiller (2000). In particular, Shiller (2000) argues that interpersonal communication, being an innate feature of humans, is more influential than traditional media or Internet. Antweiler and Frank (2004) show that high message posting is associated with greater next day volatility, suggesting that a new and modern competitor to word-of-mouth is the transmission of information through Internet. For other contributions showing evidence that word-of-mouth communications play a significant role in financial markets, see Hong, Kubik, and Stein (2004), Feng and Seasholes (2004), Brown, Ivkovic, Smith, and Weisbenner (2008), Grinblatt and Keloharju (2001), Ivkovic and Weisbenner (2005) or Cohen, Frazzini, and Malloy (2008). Other papers that emphasize the importance of social epidemics in economics are finance are Shive (2010) and Carroll (2006).

[3] See, among others, Barberis et al. (1998), Albuquerque and Miao (2010), Holden and Subrahmanyam (2002), Berk, Green, and Naik (1999), Johnson (2002), Vayanos and Woolley (2010), Makarov and Rytchkov (2009), Biais, Bossaerts, and Spatt (2010) or Wang (1993).

clarifying this debate. In particular, we acknowledge that prices do not incorporate information instantly, as in Hong and Stein (1999), but we argue that the diffusion of information must necessarily be accelerated to produce momentum, similar to the continuing overreaction effect in Daniel et al. (1998). The former effect is actually a prerequisite for prices not to be fully revealing in rational-expectations models, whereas the latter effect naturally arises from the interaction between word-of-mouth communication and price formation.

While our model shares similarities with these two models, it differs in several respects. First, Daniel et al. (1998) rely on a parsimonious and simple model, the core structure of which we borrow—yet we assume away any behavioral bias. Second, as in Hong and Stein (1999), we allow information to diffuse through the population of agents—yet, in our model this phenomenon naturally arises through word-of-mouth communication and produces two fundamentally distinct implications. First, we do not need to prevent investors from extracting information from prices. On the contrary, we *need* to allow agents to learn from prices to generate momentum—what eliminates momentum in Hong and Stein (1999) is actually the main force driving momentum in our model. Second, we do not need to postulate different types of agents *ab initio*—these types arise endogenously. In particular, we assume that agents start off being identical individuals and then let word-of-mouth communication operate to generate a rich heterogeneity of types (some agents collect more information than others). In turn, this heterogeneity leads to different, *endogenous* investment strategies: depending on their information endowments, some agents act as (rational) contrarians, while other agents act as (rational) momentum traders.

The epidemiological approach to explaining momentum and reversal developed by Hong et al. (2010) is similar to our approach. There are, however, significant differences. First, in Hong et al. (2010) agents agree to disagree and prices therefore have no informational content. Instead, in our model agents attempt to infer the information of other agents from prices. We believe that a credible theory for momentum should necessarily feature learning from prices—after all, what makes momentum a puzzle from a rational viewpoint is precisely that it can be identified in past prices and yet that it persists. Moreover, in Hong et al. (2010) information spreads like a disease—a sender transmits information to a receiver. In our paper we avoid this sender-receiver dichotomy by assuming that everyone receives information and exchanges this information at future meeting dates. On the technical side, the model of Hong et al. (2010) is difficult to solve (most of their solutions involve 3 agents), while most of our results are available in closed form, for an infinite number of agents.

Our work belongs to the information percolation literature. The term "percolation" has been introduced to finance by Duffie and Manso (2007). They borrow this term from physics and chemistry, fields in which percolation concerns the movement and filtering of fluids through porous materials. In economics, it concerns the dissemination of information through large markets. While Duffie and Manso (2007), Duffie et al. (2009), Duffie, Malamud, and Manso (2010a), and Duffie, Malamud, and Manso (2010b) focus on a setting in which markets are decentralized, our contribution is to show that information percolation also has an important role to play in centralized markets. In particular, in this literature agents meet to trade, while in our model agents meet to share information.[4]

Finally, our work originates from the "noisy rational-expectations" literature. This literature studies how prices aggregate information that is dispersedly held by agents in the economy; it recognizes that prices are a mechanism of information transmission. Paradoxically, to ensure the existence of an equilibrium, noise traders typically slow down information aggregation through prices (Grossman and Stiglitz, 1980). The informational role of prices is, therefore, necessarily imperfect, leaving room for other mechanisms of information transmission, such as word-of-mouth communication. Adding such a mechanism is our contribution to this literature.

3 Information Percolation

Grossman (1981) has proposed the concept of rational expectations equilibrium to capture the idea that stock prices aggregate information that is initially dispersed across investors. However, because of noise trading, prices aggregate information only imperfectly. This leaves room for other communication mechanisms. There are many such mechanisms, but we believe word-of-mouth communication is the one that is the most relevant for the purpose of our study. Public announcements, for instance, also contribute to spread information but, in a rational setup, public announcements do not generate trading heterogeneity. At the core of the momentum puzzle is the fact that some investors seem to follow the trend while others take the opposite position to be contrarians (Moskowitz et al., 2012), yet momentum persists. A plausible explanation of momentum must therefore preserve the concept of trading heterogeneity. This is achieved naturally through private information diffusion.

We introduce private information diffusion among investors into a noisy rational expectations equilibrium. Specifically, we offer a model of *centralized* trading and *decentralized* information gathering. Two main effects arise from our exercise. First, the intensity of information diffusion has a direct impact on the speed of the flow of information into prices. This further dictates the equilibrium price dynamics. Second, the randomness associated with

[4] Beyond the particular scope of financial markets, several papers highlight the relevance of social networks in economics. Among others, Acemoglu, Bimpikis, and Ozdaglar (2010) investigate the effectiveness of dynamics of communication as an aggregator of dispersed information and Burnside, Eichenbaum, and Rebelo (2010) show how social dynamics explain the moves on the housing market.

the information transmission mechanism that we describe below generates heterogeneity in the amount of information across agents—some become more informed than others. This further impacts trading strategies and splits the population of investors into momentum and contrarian traders.

Producing information heterogeneity with tractable social dynamics is a delicate task. For instance, Hong and Stein (1999) build a simple social network of information diffusion, but their specification precludes heterogeneity in the amount of information across agents. Similarly, Hong et al. (2010) make use of a social network borrowed from the epidemics literature which only brings about two classes of agents while the setup remains admittedly difficult to manipulate—the authors have to restrict their analysis to a network comprised of three agents only. Rather, the information percolation mechanism delivers both tractable cross-sectional dynamics and heterogeneity among agents, as we will explain in this Section.

3.1 Information Setup

We build an economy with four trading dates, indexed by $t = 0, 1, 2, 3$, and a final liquidation date, $t = 4$.[5] This economy is populated by a continuum of investors indexed by $i \in [0, 1]$. There is a risky security with payoff realized at the liquidation date. The payoff of this security, denoted $\widetilde{U}$, is *unobserved* by investors and follows a normal distribution with zero mean and precision H.[6]

Immediately prior to each trading session, each investor i obtains a private signal about the asset payoff, $\widetilde{z}_t^i$:

$$\widetilde{z}_t^i = \widetilde{U} + \widetilde{\epsilon}_t^i$$

where $\widetilde{\epsilon}_t^i$ is distributed normally and independently of $\widetilde{U}$, has zero mean, precision S, and is independent of $\widetilde{\epsilon}_j^k$ if $k \neq i$ or $j \neq t$.

The precision of private signals is constant over time and is the same across investors. We are especially keen on starting with constant and homogeneous precision, in order to perfectly isolate the effects induced by information diffusion. We describe below a mechanism through which, on top of the signals described above, agents accumulate additional signals and thus the *overall precision* of private information increases over time and becomes heterogeneous across investors. Specifically, from date 0 onward, we assume that private information diffuses across the population of agents through word-of-mouth communication.

[5] The number of four trading dates is the minimum necessary structure to show our effects; the model can be easily extended to an arbitrary number of N trading dates.

[6] We refer to the precision of a random variable as the inverse of its variance.

We formalize this idea using the setup described in Duffie et al. (2009). Starting with the large population of agents defined above, we assume that, between time 0 and time 3, each agent meets other agents randomly. Meetings take place at Poisson arrival times with intensity λ. This parameter (the *only* parameter that we add to an otherwise standard trading model) controls the intensity of social interactions in the model—it summarizes the behavior of the *whole* social network. When agents meet, they exchange truthfully their initial signals and other signals that they might have received during previous meetings (if any). Because signals are normally distributed, it is enough for an agent tells his counterparty at any meeting the total number of signals gathered so far and their average; these are sufficient statistics of the private information detained by any agent.

Now, pick two agents—say D and J—out of the crowd. D and J start with one signal. Suppose the first time they meet someone, they meet each other. They exchange their signals truthfully and therefore end up with 2 signals after the meeting. Suppose further that D meets someone else, say E who also has two signals (i.e., E also met someone before). Since D and E are part of an infinite crowd of agents, the person that E has met cannot be J, it must be someone else—meetings do not overlap.[7] Hence, after the meeting, D and E both depart with four signals each. Signals keep on adding up randomly in the exact same way for every agent in the economy.

Given the randomness of meetings, some agents may end up gathering more signals than others. This produces a natural source of heterogeneity among agents: while all agents start off being identical (with respect to the number of signals and the precision of those signals), they become heterogeneous as soon as they start meeting with each other. The theory of information percolation allows this heterogeneity to be modelled. More precisely, say that between time $t-1$ and t each agent i collects a number of ω_t^i signals, including the one received at time t, and *excluding* the signals received up to time $t-1$. That is, for the rest of this paper, when we mention the number of signals that an agent has at time t, we refer to the number of *additional* signals, thus excluding the ones collected up to time $t-1$.[8] The cross-sectional distribution of the number of signals at time t, denoted from now on by π_t, captures this percolation-driven heterogeneity. This probability density function is defined

[7] In other words, there is a zero probability that the set of agents that D has met before time t overlaps with the set of agents that J has met before time t. This eliminates the concern that we are introducing *persuasion bias* in the terms of Demarzo, Vayanos, and Zwiebel (2003): an agent might share her signals to another agents who passes those signals at subsequent meetings to other agents and maybe the same signals will come back to the first agent—without her knowledge. The infinite mass of agents prevents this double accounting of signals to happen, since the probability for an agent to meet in the future precisely those agents who got her signals is zero. Thus, for every pair (i, j) of agents, their signal sets are allways disjoint.

[8] Notice that both the distribution over the total number of signals and the distribution over additional signals may be equivalently used; we choose to use distribution of additional signals because it helps us better separate and understand the effects of information percolation on the equilibrium price and trading strategies.

to be the fraction of investors who, between $t-1$ and t, gather n signals, for any positive natural number n.

Since agents are assumed to be initially endowed with a single signal, the initial distribution of signals has 100% probability mass at $n = 1$ and 0% probability mass at $n > 1$. In the presence of information diffusion, at dates $t > 0$, the distribution π_t takes positive values for the entire set of non-zero natural numbers. For example, an agent who did not met anyone between $t-1$ and t receives only one additional signal at t and thus is of type $n = 1$. An agent who collected ten signals between $t-1$ and t receives one more signal at t and thus is of type $n = 11$, and so on. Following Duffie et al. (2009), the cross-sectional distribution of the number of signals satisfies the following Boltzmann equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi_t(n) = \lambda\pi_t * \pi_t - \lambda\pi_t = \lambda\sum_{m=1}^{n-1}\pi_t(n-m)\mu_t(m) - \lambda\pi_t(n) \tag{1}$$

where $*$ denotes the discrete convolution product and μ represents the cross-sectional distribution of the *total* number of signals, which we define in Appendix A.1. The summation term on the right hand side in (1) represents the gross rate at which new agents of a given type are created. The second term in (1) captures the rate of replacement of agents of a given type with those of some new type (e.g., after an agent of type 10 meets one of her peers, she is not going to be of the same type anymore).

Our setup has the major advantage of leading to a closed-form solution for the cross-sectional distribution over the number of additional signals. It is given in the following proposition, whose proof is provided in Appendix A.1.[9]

Proposition 1. *The probability density function over the additional number of signals collected by each agent between $t-1$ and t, with $t \geq 1$, is given by*

$$\begin{aligned}
\pi_1(n) &= e^{-n\lambda}\left(e^{\lambda}-1\right)^{n-1} \\
\pi_2(n) &= \sum_{i=0}^{n-1}\sum_{j=0}^{i}\left[\binom{i-1}{j-1}\mathbf{1}_{\{i+j=n-1\}}e^{-i\lambda-(j+1)\lambda}\left(e^{\lambda}-1\right)^{i}\right] \\
\pi_3(n) &= \sum_{i=0}^{n-1}\sum_{j=0}^{i}\sum_{k=0}^{j}\left[\binom{i-1}{j-1}\binom{j-1}{k-1}\mathbf{1}_{\{i+j+k=n-1\}}e^{-(i+j)\lambda-(k+1)\lambda}\left(e^{\lambda}-1\right)^{i}\right]
\end{aligned} \tag{2}$$

Proposition 1 provides a simple solution to the social dynamics equation (1). This simplicity is particularly compelling when considering the rich effects among agents that it is prone to generate. First, as Equations (2) show, the density function changes at each trading

[9] Although we present here closed-form solutions for the distribution of signals, we can also compute this distribution numerically through inverse Fourier transform in a very efficient fashion. We provide the details on the numerical procedure in Appendix A.1.

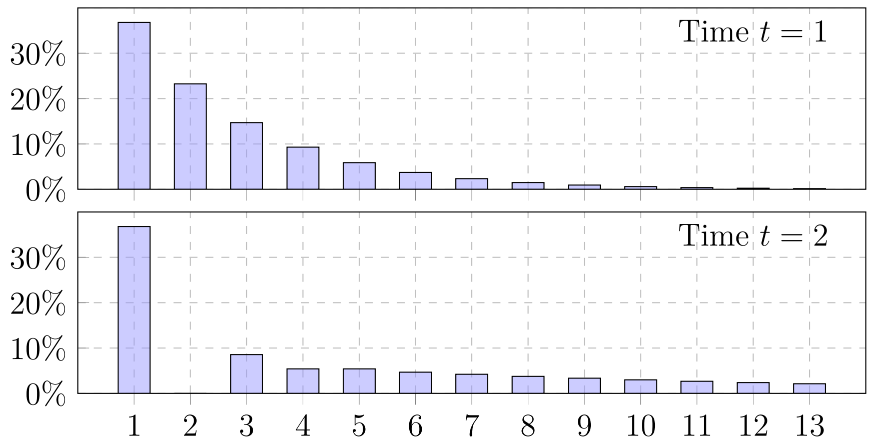

Figure 1: Evolution of Cross-Sectional Densities

The Figure depicts the evolution of the probability density function of the number of additional signals through time when agents are endowed with one signal at each period. Each graph depicts π at time $t = 1$ and $t = 2$, respectively, with $\lambda = 1$.

date. This will strongly affect the dynamics of the private information precision. Second, although agents start off at time 0 being identical with respect to the number of signals and their precision, information percolation introduces heterogeneity, which itself will change through time. All these effects have powerful implications, as we will show below.

Figure 1 illustrates the evolution of the cross-sectional distribution $\pi_t(n)$ of signals through time for a particular value of the meeting intensity λ. The aspect to retain here, particularly important to our setup, is how this distribution changes through time. The upper panel depicts the distribution at time 1, whereas the lower panel depicts the distribution at time 2. The distribution is more dispersed at time 2; even though these are additional signals and the length of time intervals is the same in both cases, between time 1 and 2 investors meet peers who have more signals accumulated through previous meetings.[10]

Before moving forward and embedding this information setup into a trading structure, one last important question must be addressed. Assume that we are between $t - 1$ and t and an agent gets from one of her peers a signal that has already been used before by the latter for the purpose of trading (a "stale" signal). Is the information provided by this particular signal still useful for trading? The answer is yes. As we will show in Section 4, only the common component of signals (i.e., the asset payoff $\widetilde{U}$) gets into the price at any trading date while, by the law of large numbers, all random components $\widetilde{\epsilon}_t^i$ are wiped out through aggregation.

[10] Other aspects are worth noting. First, the probability mass at $n = 1$ stays constant for all periods (it equals $e^{-\lambda}$, roughly 37% in this case); these are investors who do not meet anyone during the last period and consequently end up with only the usual signal received at the trading date. Second, at time 2 there is zero probability mass for $n = 2$. The reason is that no investor could gather 2 additional signals between dates 1 and 2, since that would require meeting an investor with only one signal during that period, and no such investor exists.

Consequently, agents always consider signals that they receive genuinely new and optimally choose to use them for the purpose of trading.

The next Section builds a model of trading in which information diffuses according to the percolation model outlined above.

4 Equilibrium Prices and Optimal Trading Strategies

We feed the information setup described in the previous Section into a standard model of trading, a noisy rational expectations model *à la* Grossman and Stiglitz (1980). This only departure from a standard model, while preserving the desirable insights of the noisy rational expectations approach, solves two problems at once. First, we show how the accelerated revelation of information into prices generates momentum. Second, we show how traders optimally bet on their information and separate in two fronts: trend-followers and contrarians. Momentum therefore arises and persists in an economy with rational investors.

4.1 Setup

Investors have exponential utility with common coefficient of absolute risk aversion $1/\gamma$. The asset payoff is realized and consumption takes place at time $t = 4$, while trading takes place at times $t = 0, 1, 2, 3$. At each trading date, investor i chooses the position in the risky asset, $\widetilde{D}_t^i$, to maximize expected utility of terminal wealth, denoted by $\widetilde{W}_4^i$:

$$\max_{\widetilde{D}_t^i} \mathbb{E}\left[e^{-\frac{1}{\gamma}\widetilde{W}_4^i} \middle| \widetilde{F}_t^i \right]$$

$$\text{subject to} \quad \widetilde{W}_4^i = X^i \widetilde{P}_0 + \sum_{j=0}^{2} \left[\widetilde{D}_j^i \left(\widetilde{P}_{j+1} - \widetilde{P}_j \right) \right] + \widetilde{D}_3^i \left(\widetilde{U} - \widetilde{P}_3 \right)$$

where $\widetilde{F}_t^i$ represents the information set of investor i at time t. This information set comprises: (i) private signals received at each date and collected through information percolation, as described in Section 3, and (ii) prices (endogenously determined in equilibrium and denoted by $\widetilde{P}_t$) as public signals.

Each investor i is endowed at time $t = 0$ with a quantity of the risky asset represented by X^i. The aggregate per capita supply of the risky asset at time $t = 0$, $\widetilde{X}_0 = \int_0^1 X^i di$, is normally and independently distributed with zero mean and precision Φ. New liquidity traders are assumed to enter the market in trading sessions $t = 1, 2, 3$. The incremental net supply of these traders is $\widetilde{X}_t$, a normally distributed random variable with zero mean and precision Φ.

The noisy supply assumption is a technique commonly adopted in rational expectations models to prevent asset prices from fully revealing the final payoff $\widetilde{U}$. We assume a random walk specification for the noisy supply (or, equivalently, we assume that increments in noisy supply are *iid*).[11] Under this specification, any pattern in the correlation of returns depends *only* on the time distribution of private information. It thus allows us to draw a direct link between the diffusion of information and the serial correlation of returns.[12] Moreover, since our results are not contaminated by effects arising from noise traders (who might be considered irrational), this specification further reinforce our claim for a rational explanation of momentum.[13]

Our model bears similarities with Brennan and Cao (1997), with the main difference that we embed an information diffusion mechanism. We focus on a single asset economy, featuring several trading dates and a final liquidation date, in order to keep the setup comparable with leading momentum theories, such as Daniel et al. (1998) and Hong and Stein (1999). Further differences with the literature will be discussed during the analysis below.

The solution method for finding a linear, partially revealing rational expectations equilibrium is standard and fully explained in Appendix A.3. We describe below this equilibrium.

4.2 Equilibrium with Information Percolation

Let us first introduce notation and terminology for further use. Consider an investor i who starts with one private signal at $t = 0$. As in Section 3, the number of additional private signals in periods $t = 1, 2, 3$ is denoted by ω_t^i. We do not need to keep track of each of these signals; from Gaussian theory, it is sufficient to take their average and consider it as a single signal with precision $S\omega_t^i$. We denote this aggregate signal $\widetilde{Z}_t^i$:

$$\widetilde{Z}_t^i = \widetilde{U} + \widetilde{\varepsilon}_t^i, \quad \text{where } \widetilde{\varepsilon}_t^i \sim N\left(0, \frac{1}{S\omega_t^i}\right)$$

For convenience, we group all the information concerning the random variables in the present setup in Table 1.

An important statistic for the present economy is the cross-sectional average of the number of additional signals at time t, $\Omega_t \equiv \sum_{\omega_t \in \mathbb{N}} \pi_t(\omega_t)\, \omega_t$, where $\pi_t(\omega_t)$ has been defined and

[11] *i.i.d.* incremental changes in noisy supply are likely to happen when time between consecutive trading dates is small.

[12] Other specifications, such as an an AR(1) noise trading process, gives qualitatively similar results.

[13] Liquidity, or noise traders, can actually be rational, as Cespa and Vives (2012) point out. They can be considered risk-averse, competitive hedgers, who receive a random shock to their endowment. Upon receiving that shock, these infinitely risk-averse hedgers get rid of the endowment shock in the market place, producing a similar effect that characterizes a model with noise traders. This would also bring our model closer to Moskowitz et al. (2012), who find that speculators profit from momentum strategies at the expense of hedgers.

Variable	Symbol	Mean	Precision	Comments
Fundamental	$\widetilde{U}$	0	H	
Per capita supply	$\widetilde{X}_0 = \int_0^1 X^i di$	0	Φ	at date $t = 0$
Liquidity traders	$\widetilde{X}_t$	0	Φ	at date $t = 1, 2, 3$
Private signals	$\widetilde{Z}_0^i = \widetilde{U} + \widetilde{\varepsilon}_0^i$	(for $\widetilde{\varepsilon}_0^i$) 0	S	at date $t = 0$
Private signals	$\widetilde{Z}_t^i = \widetilde{U} + \widetilde{\varepsilon}_t^i$	(for $\widetilde{\varepsilon}_t^i$) 0	$S\omega_t^i$	at date $t = 1, 2, 3$

Table 1: Random variables in the present model

computed in Section 3. Since at time 0 all investors start by receiving a single private signal, we have $\omega_0^i = \Omega_0 = 1$, $\forall i \in [0, 1]$.

We are ready to define some important variables from Theorem 1 below, typically needed for the description of the equilibrium solution. The conditional precision of agent i about the final payoff $\widetilde{U}$, given all available information (private and public signals), is denoted by K_t^i and defined in (5). The cross-sectional average of conditional precisions over the entire population of agents is denoted by K_t and defined in (6). The conditional expectation of agent i about the final payoff $\widetilde{U}$ is denoted by $\widetilde{\mu}_t^i$ and defined in (7). Two statistics help us describe the heterogeneity generated by information diffusion, both of them defined in (8). First, a_t^i represents the *marginal* informational (dis)advantage arising from private signals gathered at time t only. Second, A_t^i represents the *cumulative* informational (dis)advantage arising frm private signals gathered up to time t. These informational (dis)advantages are relative measures, comparing the individual precision of investor i with the average precision across the population of investors (marginal and cumulative). Finally, the public signals, represented by prices can be normalized under the form (9), which will prove to be extremely useful for future calculations.

Theorem 1, the proof of which is given in Appendix A.3, describes the risky asset prices and investor asset demands at each date in a noisy rational expectations equilibrium with information percolation.

Theorem 1. *There exists a partially revealing rational expectations equilibrium in the 4 trading session economy in which the risky asset price, $\widetilde{P}_t$, and individual asset demands, $\widetilde{D}_t^i$, for $t = 0, .., 3$, are given by:*

$$\widetilde{P}_t = \frac{K_t - H}{K_t}\widetilde{U} - \sum_{j=0}^{t} \frac{1 + \gamma^2 S \Omega_j \Phi}{\gamma K_t}\widetilde{X}_j \tag{3}$$

and

$$\widetilde{D}_t^i = \gamma \left[\sum_{j=0}^{t} \left(S\omega_j^i \widetilde{Z}_j^i - S\Omega_j \widetilde{Q}_j \right) - A_t^i \widetilde{P}_t \right] \tag{4}$$

where

$$K_t^i \equiv \text{Var}^{-1} \left[\widetilde{U} \middle| \widetilde{F}_t^i \right] = H + \sum_{j=0}^{t} S\omega_j^i + \sum_{j=0}^{t} \gamma^2 \Phi S^2 \Omega_j^2 \tag{5}$$

$$K_t \equiv \sum_{\omega \in \mathbb{N}} K_t^i(\omega)\pi_t(\omega) = H + \sum_{j=0}^{t} S\Omega_j + \sum_{j=0}^{t} \gamma^2 \Phi S^2 \Omega_j^2 \tag{6}$$

$$\widetilde{\mu}_t^i \equiv \mathbb{E} \left[\widetilde{U} \middle| \widetilde{F}_t^i \right] = \frac{1}{K_t^i} \sum_{j=0}^{t} \left[S\omega_j^i \widetilde{Z}_j^i + \gamma^2 S^2 \Phi \Omega_j \widetilde{Q}_j \right] \tag{7}$$

$$a_t^i \equiv S\omega_t^i - S\Omega_t, \;\; A_t^i \equiv K_t^i - K_t \tag{8}$$

$$\widetilde{Q}_t \equiv \widetilde{U} - \frac{1}{\gamma S \Omega_t} \widetilde{X}_t \tag{9}$$

The optimal trading strategy of the individual investor i at time $t = 1, 2, 3$, denoted by $\Delta\widetilde{D}_t^i \equiv \widetilde{D}_t^i - \widetilde{D}_{t-1}^i$ takes the following form:

$$\gamma \left[a_t^i \left(\widetilde{Z}_t^i - \widetilde{P}_{t-1} \right) + S\Omega_t \left(\widetilde{Z}_t^i - \widetilde{P}_{t-1} \right) - \frac{K_t}{1 + \gamma^2 S \Phi \Omega_t} \left(\widetilde{P}_t - \widetilde{P}_{t-1} \right) - A_t^i \left(\widetilde{P}_t - \widetilde{P}_{t-1} \right) \right] \tag{10}$$

4.2.1 Interpretation of Results without Information Percolation

First, we break down the results of Theorem 1 in an economy without information percolation ($\lambda = 0$). In this case we have $\omega_t^i = \Omega_t = 1$, $a_t^i = A_t^i = 0$, and $K_t^i = K_t$, for any $i \in [0, 1]$ and for $t = 0, 1, 2, 3$.

Equation (3) shows that the asset price is a linear function of the final payoff and supply shocks, as is customary in the noisy rational expectations literature. Equation (4) shows that the optimal demand of investor i in period t depends on private signals ($\widetilde{Z}_j^i$) and public signals ($\widetilde{Q}_j$). Without information percolation, this optimal demand boils down to

$$\widetilde{D}_1^i = \gamma S \sum_{j=0}^{t} \left(\widetilde{Z}_j^i - \widetilde{Q}_j \right)$$

Investor i compares private signals with public signals, demands more of the risky asset if she observes relatively higher private signals ($\widetilde{Z}_j^i > \widetilde{Q}_j$), and trades more aggressively if risk aversion ($1/\gamma$) is low or the precision of private information (S) is high.

For the purpose of our study, it is useful to analyze the optimal trading strategy of investor

i at time t, stated in Equation (10). It has four terms. With no information percolation, the first and the last term are equal to zero (investors have homogeneous information endowments and thus $a_t^i = A_t^i = 0$). Only two terms drive in this case the optimal trading strategy: the second and the third term in (10). The second term, which becomes $\gamma S \left(\widetilde{Z}_t^i - \widetilde{P}_{t-1}\right)$ without information percolation, shows how investor i trades based on *private* information: if the investor observes a positive signal, higher than the past price, she increases her position in the asset. Furthermore, a lower risk aversion or a higher private signal precision amplifies this effect. The third term, which becomes $-\frac{\gamma K_t}{1+\gamma^2 S\Phi}\left(\widetilde{P}_t - \widetilde{P}_{t-1}\right)$ without information percolation, shows how investor i trades based on *public* information: if she observes a price today higher than the past price, there is a chance that supply is low today, hence the investor decreases her position in order to accommodate this tight supply.

As in Brennan and Cao (1997), it is convenient to consider the expected trade of investor i conditional on the price change at time t, $\Delta\widetilde{P}_t$. This measure describes the trading behaviour of agent i and can be computed by the econometrician, who observes neither supply shocks nor private signals.

$$\begin{aligned}
\mathbb{E}\left[\Delta\widetilde{D}_t^i \middle| \Delta\widetilde{P}_t\right] &= \gamma S \mathbb{E}\left[\widetilde{Z}_t^i - \widetilde{Q}_t \middle| \Delta\widetilde{P}_t\right] \\
&= \gamma S \mathbb{E}\left[\widetilde{U} + \widetilde{\varepsilon}_t^i - \widetilde{U} + \frac{1}{\gamma S}\widetilde{X}_t \middle| \Delta\widetilde{P}_t\right] \\
&= \mathbb{E}\left[\widetilde{X}_t \middle| \Delta\widetilde{P}_t\right] \\
&= m_t \Delta\widetilde{P}_t
\end{aligned} \tag{11}$$

where the sign of the coefficient m_t determines the investment style. When this coefficient is positive, an agent tends to buy after price increases and sell after price decreases. Thus, the econometrician observes momentum trading when m_t is positive and contrarian trading when m_t is negative.

The following Proposition shows that, in an economy with homogeneous information endowments, all investors find it optimal to be contrarians.

Proposition 2. *In an economy without information percolation, all investors adopt contrarian strategies, i.e., their trades are negatively correlated with the price change in the current period.*

Proof. Following from (3) and (11), the conditional expected trade can be written as

$$\mathbb{E}\left[\Delta\widetilde{D}_t^i \middle| \Delta\widetilde{P}_t\right] = \underbrace{-\frac{1+\gamma^2 S\Phi}{\gamma K_t}\frac{\operatorname{var}\left(\widetilde{X}_t\right)}{\operatorname{var}\left(\Delta\widetilde{P}_t\right)}}_{m_t}\Delta\widetilde{P}_t$$

It is straightforward to see that $m_t < 0$. Thus, all investors are contrarians. $\square$

Proposition 2 raises an intriguing question. If a noisy rational expectations model with homogeneous information endowments were to generate momentum in asset returns, then who would accumulate momentum profits by following the trend? According to Proposition 2, no investor will find it optimally to do so. This is not what we observe. Moskowitz et al. (2012) document that *speculators' position load positively on time series momentum, while hedgers' positions load negatively on it.* Any theory of momentum must be consistent with this observed heterogeneity in trading positions.

Thus, although it is possible for a standard rational expectations model to generate momentum (this happens only for specific paths in the private information precision, as we will show in Section 5.3), it will not explain why some traders engage in momentum trading and others in contrarian trading, for all traders in such a model are contrarians. Below we show information percolation helps our model to address this issue and to generate this prediction.

4.2.2 Interpretation of Results with Information Percolation

Although this Section does not yet elaborate about momentum, it can be regarded as the essence of our contribution to the momentum literature. It describes a credible and appealing mechanism through which investors with heterogeneous trading strategies optimally coexist into the market—some agents are momentum traders, others are contrarians. It also describes important implications for momentum timing. More precisely, we will show below how trading on momentum crucially depends on the timing of the private information.

Information percolation generates two simultaneous effects. First, when $\lambda > 0$ we have $\Omega_t > 1$, for $t = 1, 2, 3$. Thus, the average precision of private information, $S\Omega_t$, increases over time as agents accumulate more signals through private meetings. This has an impact on prices, because prices directly depend on the average precision of private information, as shown in (3). In Section 5, we will analyze in depth this effect on the autocorrelation in asset returns.

Second, information percolation generates heterogeneity in information endowments and thus some investors become better informed than others. In this case, at trading dates $t = 1, 2, 3$, the marginal and/or the cumulative informational advantages, a_t^i and A_t^i, are different from 0. This has a direct impact on optimal trading strategies; the first and last term in the optimal trading strategy (10) become relevant, in the following ways.

Consider an investor who has a *marginal* informational advantage at time t, i.e., $a_t^i > 0$ and who observes a private signal relatively higher than the past price. The first term in the optimal trading strategy, $\gamma a_t^i(\widetilde{Z}_t^i - \widetilde{P}_{t-1})$, shows that this investor increases her position in the risky asset, adding up to the position coming from the second term in (10). Put differently, because the investor has a marginal informational advantage, she feels more confident about her private signals received at time t, and thus trades more aggressively on private information.

Consider an investor who has a *cumulative* informational advantage at time t, i.e., $A_t^i > 0$ and who observes a price today higher than the past price. According to the last term in (10), that is, $-\gamma A_t^i \left(\widetilde{P}_t - \widetilde{P}_{t-1}\right)$, the investor decreases her position in the risky asset, adding up to the (already negative) position coming from the third term in (10). Put differently, because the investor has an overall information advantage, she is better able to deduce the supply shock from the price signal and to accommodate it, and thus trades more aggressively on public information.

It follows that investors with marginal and cumulative information advantages trade more aggressively than the average investor, both on private and public information. But it is not clear yet if these adjustments in the trading strategy make the investor a trend follower or a contrarian—the first two terms in (10) represent momentum trading,[14] whereas the last two terms in (10) represent contrarian trading. Depending on which effect dominates, the investor optimally chooses to be a contrarian or trend-follower. The converse applies for an investor with marginal and cumulative information *disadvantages*.

Figure 2 helps us to be explicit about the tradeoff between being a trend-follower or a contrarian and illustrates the heterogeneity in trading strategies generated by information percolation. The Figure depicts the coefficient m_1 for different values of the meeting intensity λ (the analysis is simpler at time $t = 1$ because the marginal and cumulative informational (dis)advantages are equal). According to Proposition 2, for $\lambda = 0$ all investors are contrarians. When λ is positive, however, investors adopt heterogeneous trading strategies. Consider three investors types: (*i*) the 5% percentile *less informed investor* according to the probability density function described in Section 3 (dotted red line), (*ii*) the *average informed investor* (black solid line), and (*iii*) the 95% percentile *better informed investor* (dashed blue line). Accordingly, the shaded area between the 5% percentile and 95% percentile represent 90% of the investor population. Figure 2 shows that, for higher values of λ, the relatively well informed investors become momentum traders (dark gray area in the graph). The marginal informational advantage is high enough for them to trade aggressively on their private information.

At this point, an important difference arises between our explanation of momentum and the one provided by Hong and Stein (1999). Like Hong and Stein (1999), we recognize that a credible theory for momentum must necessarily imply momentum traders, but unlike

[14] It can be easily shown that $\widetilde{Z}_t^i - \widetilde{P}_{t-1}$ is positively correlated with $\Delta\widetilde{P}_t$.

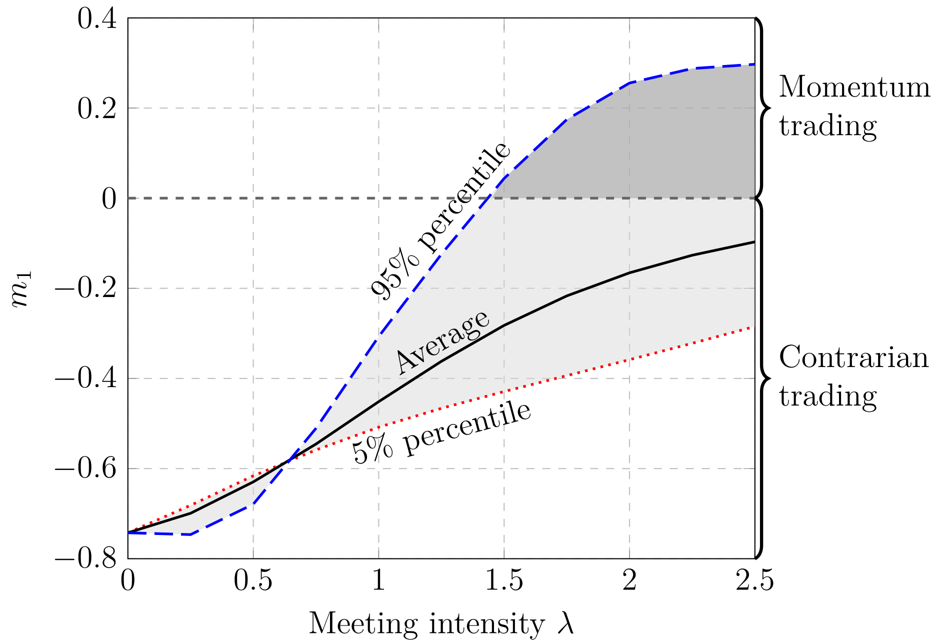

Figure 2: Information Percolation and Heterogeneous Trading Strategies at $t = 1$
The Figure depicts the coefficient of the price difference $\Delta\widetilde{P}_t$ in the expectation $\mathbb{E}\left[\Delta\widetilde{D}^i_1\middle|\Delta\widetilde{P}_1\right]$ at time $t = 1$, i.e., m_1. A positive coefficient means momentum trading, whereas a negative coefficient means contrarian trading. There are three lines. The black solid line corresponds to the average informed investor, the red dotted line corresponds to the 5% percentile less informed investor, and the blue dashed line corresponds to the 95% percentile better informed investor. Thus, the gray area represents 90% of the population of investors. The light gray area represent the contrarian investors ($m_1 < 0$), whereas the dark gray area represent the momentum investors ($m_1 > 0$).

Hong and Stein (1999) we do not need to postulate trading types *ab initio*—they arise endogenously through word-of-mouth communication. More important, although one can arguably say that Hong and Stein (1999) also have an information diffusion mechanism in their setup, we bring forth an information diffusion mechanism which *naturally* generates heterogeneous information endowments and thus heterogeneous trading strategies, while preserving rationality.

The analysis is more involved at time 2, because the marginal and informational advantages need not be equal anymore (some traders can have a marginal advantage but not a cumulative one, or vice-versa). It turns out that this extra layer of analysis adds depth to our understanding of momentum trading. Figure 3 illustrates this. Consider first an investor who was less informed at time 1 (panel *a* of Figure 3). If λ is high, there is a high chance for a large marginal advantage at time 2. Consequently most investors who had a low information advantage at time 1—and thus were probably contrarian traders—adopt trend following strategies at time 2. On the contrary, an investor who was well informed at time 1 (panel

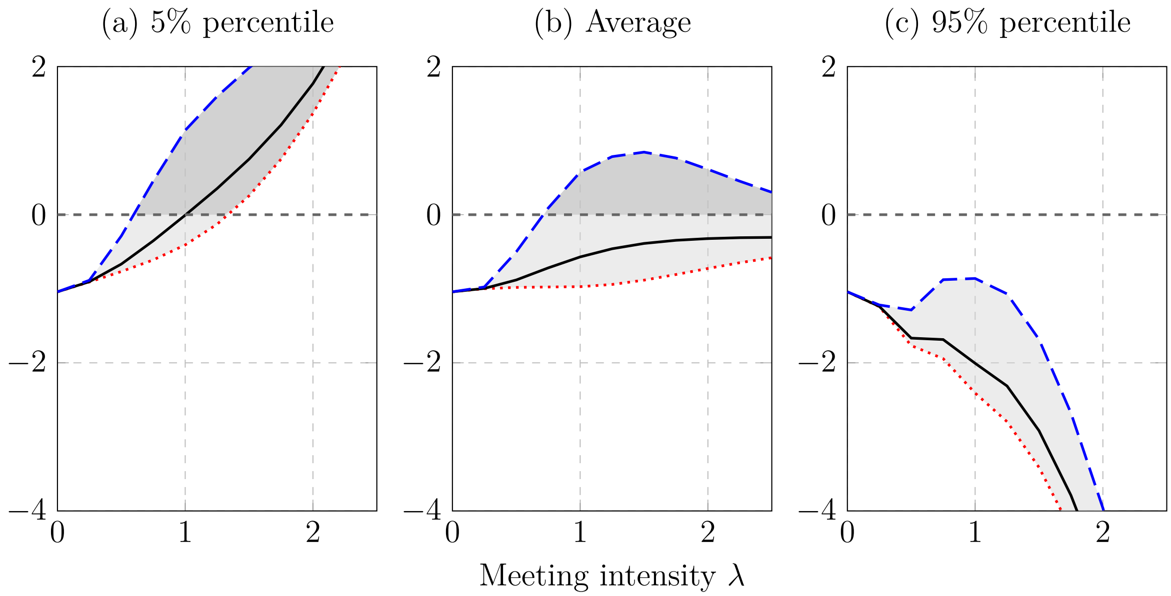

Figure 3: Information Percolation and Heterogeneous Trading Strategies at $t = 2$ The Figure depicts the same coefficient as in Figure 2, at time 2, i.e., m_2. The lines are the same as in Figure 2. Panel (a) is for the trader who at time 1 was the 5% less informed investor, panel (b) is for the trader who at time 1 was the average investor, and panel (c) is for the trader who at time 1 was the 95% better informed investor.

c of Figure 3)—and who was a momentum trader—becomes certainly a contrarian at time 2. This happens because her cumulative advantage overwhelms the marginal informational advantage and thus the investors prefer to accommodate noise trading and be a contrarian.

Thus, investors who obtain "hot information" (in the context of our model, investors who have a significant *marginal* informational advantage) optimally choose to trade aggressively on their private information. An econometrician who analyzes the trading strategies of such investors deduces that they bet on momentum, much as Moskowitz et al. (2012) find that speculators are the ones that profit from momentum. On the other hand, investors who have a significant *cumulative* informational advantage (who can be seen as investors with a broad historical knowledge of the market) place more aggressive bets on public information. An econometrician who analyzes their trading strategies deduces that they are contrarians, much as Moskowitz et al. (2012) find that commercial investors, or hedgers, are contrarians. The bottom line is that information percolation generates heterogeneous information endowments and both momentum and contrarian investors coexist and trade in the same market.

This also suggests that investors who have a large marginal advantage early on and a large cumulative advantage later optimally chose to "ride the bubble" first and bet against it near the end of the run-up. On the contrary, it is less profitable to bet on momentum near the end of the run-up.

It is not only the distinction between trading strategies that matters, but also the proportion of investors who adopt various trading strategies. The proportion of momentum traders, outlined by dark gray areas in Figures 2 and 3, appears to move through time. As such, our model may alternatively be viewed as one with a time-varying representative agent.[15] More to the point, the proportion of momentum traders at any point in time should have an impact on the dynamics of asset returns. We expect a higher proportion of momentum traders to be accompanied by momentum in asset returns, and it turns out that this is exactly the case. We turn now to this implication.

5 Information Percolation and Momentum

The momentum effect of Jegadeesh and Titman (1993) presents an important challenge to theoretical explanations: if momentum is a *profitable* empirical anomaly that is easily identifiable from past returns, why does it persist? We show that, in our rational-expectations equilibrium model, returns exhibit momentum, some agents trade on it, and yet momentum persists. It is therefore possible to build a rational theory in which momentum appears as a profitable empirical anomaly. We explain below how word-of-mouth communication naturally generates these results and we highlight the main differences with alternative theories in the literature.

5.1 Momentum

We want to understand how social interactions generate momentum. To do so, we proceed as in Banerjee, Kaniel, and Kremer (2009) and compute

$$E[\Delta\widetilde{P}_t|\Delta\widetilde{P}_{t-1}] = \frac{\text{cov}(\Delta\widetilde{P}_t, \Delta\widetilde{P}_{t-1})}{\text{var}(\Delta\widetilde{P}_{t-1})}\Delta\widetilde{P}_{t-1} \equiv \rho_t\Delta\widetilde{P}_{t-1} \tag{12}$$

where the coefficient ρ_t, whose formula is given in Appendix A.4, dictates whether returns exhibit momentum or reversal. That is, by regressing next-period returns on last-period returns, one can easily see whether returns exhibit momentum ($\rho > 0$) or reversal ($\rho < 0$).

We first shut down word-of-mouth communication among agents and obtain the result highlighted in the proposition below, whose proof is given in Appendix A.4.

Proposition 3. *In the absence of word-of-mouth communication among investors, returns exhibit reversal.*

[15]We thank Sergio Rebelo por pointing out this interpretation.

Reversal commonly obtains in standard rational-expectations models (e.g., Wang (1993)). This result is driven by what Cespa and Vives (2012) call the "Keynesian" effect: investors speculate on short-term price swings induced by noise trading, a situation likely to arise when information diffuses slowly. To illustrate this, suppose most investors seldom receive fundamental information; their best move is then to speculate on idiosyncratic price shocks—in our model, noise trading. For instance, if the demand for the asset is high and so is its price, investors can accommodate the demand by selling the stock today, expecting to buy it back at a lower price when demand reverts. In a market populated by such *contrarian* investors returns tend to exhibit reversal. And, as shown in Proposition 2, in an economy without information percolation, *all* investors follow such contrarian strategies. As such, standard rational expectation models returns usually exhibit reversal, a conclusion also reached by Banerjee et al. (2009), who show that momentum *never* obtains.

We will now show that word-of-mouth communication generates both a large fraction of momentum traders—overturning the "Keynesian effect" described in Cespa and Vives (2012)—and a sustained increase in average market precision over time—a feature dismissed in Banerjee et al. (2009). To show this, we plot the serial correlation for the first- (the solid blue line) and the second-period returns (the dashed red line) in the top panel of Figure **??**.[16]

When the meeting intensity is low returns exhibit reversal and all investors are contrarians; this is the result that we previously highlighted. As social interactions intensify, however, the flow of information accelerates, the average precision of private information rises, and returns eventually exhibit momentum. This effect is closely related to the fraction of momentum traders in the economy (the bottom panel of Figure **??**). In particular, the range of meeting intensities for which momentum obtains can be split into two parts. First, for an intermediate range of intensities ($0.5 < \lambda < 1$), momentum arises even though all investors are contrarians—this effect is solely triggered by the increase in precision. Second, for larger meeting intensities ($\lambda > 1$), a fraction of momentum traders emerges—as momentum gets stronger, some traders find it profitable to bet on it. Finally, the magnitude of momentum ultimately decreases as more and more investors trade on it. Overall, our model predicts a hump-shaped relation between momentum and the speed at which information diffuses; this is the empirical pattern documented by Hong et al. (2000).

A gradual increase in agents' precision of private information is the mechanism producing momentum in Holden and Subrahmanyam (2002) and Cespa and Vives (2012). What makes our rational explanation for momentum unique is that word-of-mouth communication *endogenously* produces this increase in precision. Not only does it increase the average precision in

[16] It is straightforward to show that the sign of the coefficient ρ_t in (12) is the same as the sign of the correlation between two consecutive returns.

the economy, but it also generates heterogeneity in trading strategies; a fraction of traders will find it optimal to bet on momentum. Combining these two elements represents our main contribution: we build a model in which rational investors trade on momentum and, yet, in which momentum persists.

To understand how momentum and momentum trading coexist in a rational setup, we now show that momentum can be seen both as an information effect and as a risk effect. We argue that both effects are two sides of the same coin; in the sequel, we present each effect in turn.

5.1.1 Momentum as an Information Effect

Prices in (3) reflect both fundamental information (the fundamental value of the asset) and idiosyncratic risk (noise trading). In this section, we focus on fundamental information. To do so, suppose that we know the fundamental value $\widetilde{U}$ of the asset (say it is $\widetilde{U} = 1$); we can then compute how prices converge to the fundamental on average

$$E[\widetilde{P}_t | \widetilde{U}] = \left(1 - \frac{H}{K_t}\right) \widetilde{U} \equiv T_t \widetilde{U}.$$

The coefficient T reflects the fundamental trend in prices; as the meeting intensity goes to infinity, $T = 1$ and prices match the fundamental on average. We plot the fundamental trend T in Figure 4.

When social interactions are moderate (the black solid line), prices slowly converge to the fundamental and the fundamental trend is weak. As social interactions intensify, the flow of information through prices accelerates and the fundamental trend gets stronger and stronger. A more and more pronounced trend generates momentum (which is *observable* in the prices by everyone) and encourages some investors to chase it. If these momentum traders observe a price increase today, they follow the trend by buying additional units of the stock. Doing so, they increase the demand for the asset and push prices further up. We emphasize that investors are rational and use all available information, including prices.

This last point is one of the main elements that distinguishes our model from behavioral explanations, in particular those of Hong and Stein (1999) and Daniel et al. (1998). On the one hand, our explanation helps reconcile both theories: to produce momentum, we need prices not to reflect information instantly—the *under-reaction* explanation of Hong and Stein (1999)—but this information must necessarily be accelerated (through word-of-mouth communication)—in the same vein with the *continuing over-reaction* explanation of Daniel et al. (1998).[17] On the other hand, our explanation critically relies on rational learning from prices: unlike

[17] Note that investors do not over-react to information in our setup.

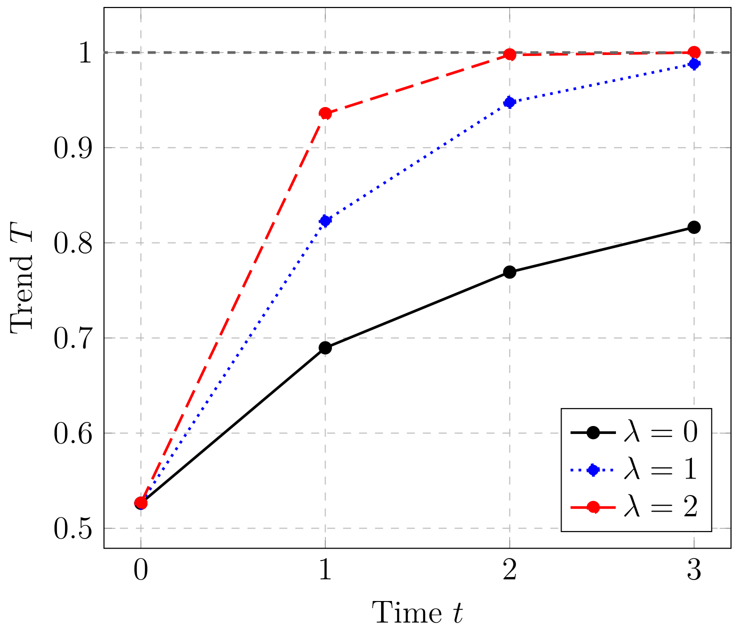

Figure 4: Fundamental Trend in Prices

The Figure depicts the fundamental trend in prices over time. The fundamental trend is plotted for a meeting intensity of $\lambda = 0$ (the black solid line), $\lambda = 1$ (the blue dotted line), and $\lambda = 2$ (the red dashed line). The calibration is $H = S = \Phi = 1$ and $\gamma = \frac{1}{3}$.

Hong and Stein (1999) who preclude learning from prices and postulate momentum traders *ab initio*, we need that investors rationally process information from prices to *endogenously* generate momentum and momentum trading. What eliminates momentum in Hong and Stein (1999) is actually the main force driving momentum in our model.

Our rational explanation is also distinct from that of Makarov and Rytchkov (2009) and Albuquerque and Miao (2010). Using a rational-expectations equilibrium model, Makarov and Rytchkov (2009) show that the slow diffusion of information may produce momentum, an under-reaction phenomenon. In our model, instead, we need that information be accelerated to produce momentum. Albuquerque and Miao (2010) construct a dynamic model in which informed investors receive "advanced" information about the fundamental. Prices adjust to reflect this information and keep moving in the same direction as this information is further corroborated by the arrival of public information. In their model, information is instantly incorporated into prices and momentum arises as investors under-react to this information. In our model, prices do not reflect information immediately and the flow of information needs to be accelerated to produce momentum.

In a general equilibrium framework, markets must clear. As a result, if one agent goes long, another agent must necessarily take the other side of the trade, i.e., if some agents are momentum traders, some other agents must necessarily be contrarians. But, if word-of-mouth communication generates momentum and that momentum is identifiable from prices, why

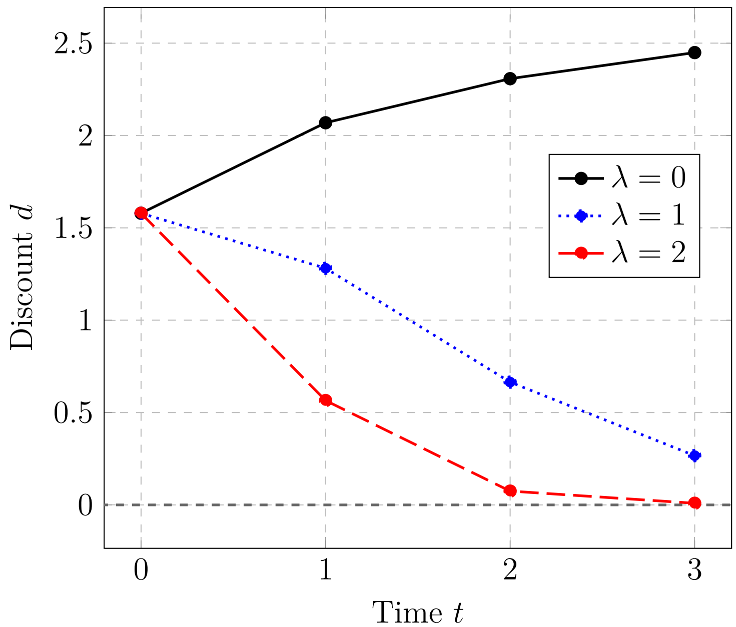

Figure 5: Price Discount and Compensation for Risk

The Figure depicts the price discount over time. The price discount is plotted for a meeting intensity of $\lambda = 0$ (the black solid line), $\lambda = 1$ (the blue dotted line), and $\lambda = 2$ (the red dashed line). The calibration is $H = S = \Phi = 1$ and $\gamma = \frac{1}{3}$.

should some agents be contrarians? To answer this question, we take an alternative standpoint and look at the effect of word-of-mouth communication on risk.

5.1.2 Momentum as a Risk Effect

The second part of prices in (3) relates to noise trading shocks: following the interpretation of Wang (1993), it represents a discount on the price that increases (conditional) expected future returns to compensate investors for bearing noise trading risk. In this section, we concentrate on this part. Suppose that we observe the sequence $\{\widetilde{X}_j\}_{j=0}^t$ of noise trading shocks; we can then compute how the price discount evolves on average:

$$E[\widetilde{P}_t|\{\widetilde{X}_j\}_{j=0}^t] = -\frac{1}{\gamma K_t}\sum_{j=0}^{t}(1+\gamma^2 S\Omega_j\Phi)\widetilde{X}_j \equiv -\sum_{j=0}^{t} d_{t,j}\widetilde{X}_j$$

Say each supply increment is $\widetilde{X}_t = 1$. Then, the price discount at time t equals $d_t = \sum_{j=0}^t d_{t,j}$. When the meeting intensity goes to infinity, the price discount d goes to zero and the compensation for noise trading risk vanishes. We plot the price discount in this particular case ($\widetilde{X}_t = 1$) in Figure 5.

When the meeting intensity is low, the price discount keeps increasing over time. Because there is not much information, it becomes more and more risky to hold the asset and the compensation for noise trading risk goes up. As information percolates more intensely, the flow of information accelerates, and more and more information gets impounded into the price. As a result, it becomes less and less risky to hold the asset, the compensation for noise trading risk goes down, and so does the price discount. This effect on the price discount generates momentum (since the price discount decreases over time, prices keep increasing on average, producing momentum in stock returns).

That the risk associated with holding the asset decreases over time also explains why some investors implement contrarian strategies although returns exhibit momentum. While social interactions produce momentum encouraging speculation on long-term information (the fundamental value of the asset), they also make it less risky for investors to bet on price swings (short-term speculation on noise trading demand). Since short-term speculation gets less and less risky, some investors optimally decide to accommodate noise trading demand, pursuing a contrarian strategy. Hence, in our model, some agents are momentum traders, others are contrarians, and momentum subsists.

This interpretation of momentum in terms of risk further distinguishes our explanation from other rational-expectations theories, in particular that of Biais et al. (2010). Biais et al. (2010) build a dynamic model in which momentum may arise depending on the relative persistence of the fundamental and noise trading risk.[18] Instead, in our model, persistence arises endogenously through the acceleration of information brought about by word-of-mouth communication among investors.

5.2 Momentum Profits

Any investor who has access to past prices can make profits on momentum—no further knowledge is needed. What makes momentum a puzzle is precisely that momentum trading is profitable *ex-ante* and yet momentum persists. A credible explanation for momentum is therefore one in which momentum is observable and the econometrician—an individual who observes the tape, but who has no further knowledge of the price formation, nor the information structure—can make profits on it.

To show that the econometrician makes momentum profits in our model, we compute the expected cumulative profits of a momentum strategy:

$$\text{Momentum Profits } = \mathbb{E}\left[\sum_{t=1}^{4}\left(\mathbf{1}_{\Delta\widetilde{P}_t>0}-\mathbf{1}_{\Delta\widetilde{P}_t<0}\right)\Delta\widetilde{P}_t\right]$$

In words, we compute the cumulative profits of a strategy that starts at time 1 and that

[18]Other rational theories involving persistent cashflows include Berk et al. (1999) and Johnson (2002).

consists in buying one unit of the stock at time t if prices went up ($\Delta\widetilde{P}_t > 0$) or selling one unit of the stock if prices went down ($\Delta\widetilde{P}_t < 0$).

To benchmark the profits of the econometrician, we consider another agent who—like the econometrician—only observes prices, but who—unlike the econometrician—has common knowledge about the structure of the economy (i.e., the agent knows the structure of the economy, as described in Theorem 1). We denote this agent by c (for *common*). Since agent c has no private information ($\omega_t^c = 0\ \forall t$), nor an informational advantage ($A_t^c = 0\ \forall t$), it follows from (4) that her portfolio satisfies

$$\widetilde{D}_t^c = \sum_{j=0}^{t} \widetilde{X}_j - S\gamma \sum_{j=0}^{t} \Omega_j \widetilde{U}. \tag{13}$$

Following this strategy, agent c's expected cumulative profits are given by

$$\mathbb{E}\left[\sum_{t=1}^{4} \widetilde{D}_t^c \Delta\widetilde{P}_t\right] = \frac{1}{\gamma\Phi}\sum_{t=0}^{4} t\left(\frac{1}{K_t} - \frac{\mathbf{1}_{\{t<4\}}}{K_{t+1}}\right).$$

In words, agent c makes profits on the improvement in the average market precision from one date to the other.

Profits on both strategies may be considered as profits in excess of those of a buy-and-hold strategy, since a buy-and-hold strategy generates null expected profits in our setup.[19] We plot econometrician's profits (the solid red line) and agent c's profits (the dashed blue line) in Figure 6. The difference between the red and the blue lines represents the marginal benefit of knowing the structure of the economy.

For low meeting intensities, the econometrician's strategy performs poorly since returns exhibit reversal. Instead, agent c's strategy generates large profits: because the average precision Ω is low, other agents in the market are not significantly more informed than agent c; this allows agent c to make large profits at the expense of noise traders, the first term in (13). Hence, when information diffuses slowly, the strategy of a rational agent (e.g., agent c) outperforms a naive momentum strategy.

As information percolation accelerates, the profits on both strategies rapidly converge: more and more agents become significantly more informed than agent c, impairing her ability to make profits on noise traders. As a result, her profits gradually shrink and ultimately

[19] A buy-and-hold strategy generates

$$\mathbb{E}[P_4 - P_1] = \left(\frac{1}{K_1} - \frac{1}{K_4}\right)\left(H\mathbb{E}[\widetilde{U}] + \frac{1}{\gamma}\sum_{j=0}^{1}(1+\gamma^2 S\Omega_j\Phi)\mathbb{E}[\widetilde{X}_j]\right) - \frac{1}{\gamma K_4}\sum_{j=2}^{4}(1+\gamma^2 S\Omega_j\Phi)\mathbb{E}[\widetilde{X}_j] = 0$$

since $\widetilde{U}$ and $\{\widetilde{X}_j\}_{j=0}^4$ have mean 0.

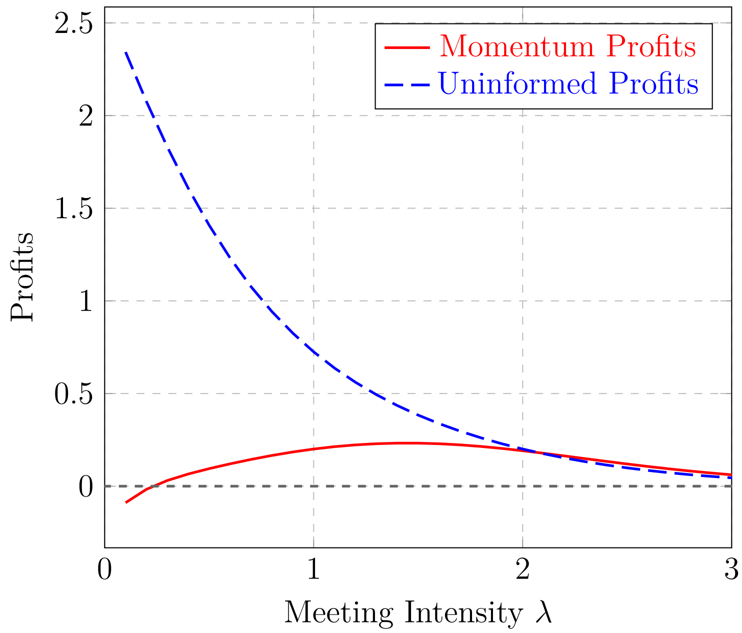

Figure 6: Profits on a Momentum Strategy as a Function of λ
The solid red line represents the profits on a momentum strategy as a function of the meeting intensity λ. The dashed blue line represents the profits of an investors equally informed as the econometrician, but who also knows the structure of the economy. The calibration is $H = S = \Phi = 1$ and $\gamma = \frac{1}{3}$.

match those of a naive momentum strategy.

This last observation shows how social interactions offer a credible explanation for momentum: social interactions increase the average market precision over time causing some investors to chase the trend; but this trend-chasing behavior is precisely what allows momentum to arise and, by the same token, what allows momentum strategies to be profitable. In our model, momentum is observable and some agents trade on it.

5.3 Alternative Model with Exogenous Increase in Precision

A gradual increase in average market precision is sufficient to generate momentum (see, e.g., Holden and Subrahmanyam, 2002); it is therefore important to understand how our explanation differs from one in which the average precision is assumed—exogenously—to increase over time. To do so, we assume away word-of-mouth communication among investors and propose a model in which investors' signal precision S_t arbitrarily improves over time ($S_t > S_{t-1}$). We present the resulting equilibrium in Theorem 2.

Theorem 2 (**An economy with exogenous increase in information precision).** *There exists a partially revealing rational expectations equilibrium in the 4 trading session economy in which the risky asset price,* $\widetilde{P}_t$, *and individual asset demands,* $\widetilde{D}_t^i$, *for* $t = 0, .., 3$, *are given* *by:*

$$\widetilde{P}_t = \frac{K_t - H}{K_t}\widetilde{U} - \sum_{j=0}^{t}\frac{1+\gamma^2 S_j \Phi}{\gamma K_t}\widetilde{X}_j$$

and

$$\widetilde{D}_t^i = \gamma \sum_{j=0}^{t} S_j \left(\widetilde{Z}_j^i - \widetilde{Q}_j\right)$$

where

$$\widetilde{Q}_t \equiv \widetilde{U} - \frac{1}{\gamma S_t}\widetilde{X}_t$$

$$K_t \equiv \mathrm{Var}^{-1}\left[\widetilde{U}\middle|\widetilde{F}_t^i\right] = H + \sum_{j=0}^{t} S_j + \sum_{j=0}^{t}\gamma^2\Phi S_j^2$$

$$\widetilde{\mu}_t^i \equiv \mathbb{E}\left[\widetilde{U}\middle|\widetilde{F}_t^i\right] = \frac{1}{K_t}\sum_{j=0}^{t}\left[S_j\widetilde{Z}_j^i + \gamma^2 S_j^2 \Phi \widetilde{Q}_j\right]$$

The optimal trading strategy of the individual investor i at time $t = 1, 2, 3$ is given by

$$\Delta\widetilde{D}_t^i \equiv \widetilde{D}_t^i - \widetilde{D}_{t-1}^i = \gamma S_t \left(\widetilde{Z}_t^i - \widetilde{Q}_t\right)$$

Proof. See Brennan and Cao (1997).

Applying the measure of momentum of Banerjee et al. (2009) to this particular setup, we obtain the following result.

Proposition 4. *In an economy with exogenous increase in information precision, momentum arises at time $t = 1$ if and only if agent's precision shows sufficient improvement over time.*

Proof. Substituting $S\Omega_t \equiv S_t$ in (42) and simplifying shows that returns exhibit momentum at time $t = 1$ if

$$S_1 - S_0 > \frac{H}{1 + \gamma^2 \Phi S_0} > 0.$$

The right-hand side is strictly positive. ■

To generate momentum, the necessary improvement in agents' precision is increasing in the precision H of the fundamental and risk aversion $1/\gamma$ and decreasing in the precision of the supply increments Φ. Furthermore, if agents' past precision of private information is large (S_0 large), a large precision of private information today (S_1 large) is needed to produce momentum. To illustrate this, suppose agent's precision increases geometrically over time, i.e.,

$S_t = a^t$. Using our calibration, we need at least $a \approx 2$ to generate momentum—the precision of the signal at time $t = 1$ must be at least twice that of the signal at time $t = 0$.

This observation allows us to highlight the first contribution of our model with respect to a model with exogenous increase in average precision: while a substantial improvement in agent's precision may be difficult to justify exogenously, this phenomenon naturally arises when investors interact through word-of-mouth communication. Our model therefore provides a micro-foundation for this gradual increase in agents' precision.

The second and most important contribution of our model becomes immediately apparent when considering agents' trading strategy. Proposition 5, directly related to Proposition 2, makes this point.

Proposition 5. *In an economy with exogenous increase in information precision, all investors implement contrarian trading strategies at time $t = 1$.*

Proof. In this setting, agents have no informational advantage ($A_t^i \equiv a_t^i \equiv 0$), since they all have the same precision. As a result, the trading measure of section 4 simplifies to

$$\text{cov}(\Delta\widetilde{D}_t^i, \Delta\widetilde{P}_t) \equiv -\frac{1 + \gamma^2 S_1 \Phi}{\gamma \Phi K_0} < 0$$

This expression is negative for all admissible parameter values. ■

An exogenous increase in average market precision can generate momentum with contrarian investors only. But what makes momentum a puzzle is precisely that some agents trade on it and yet that it persists. An exogenous increase in average precision therefore cannot provide a satisfactory answer to the momentum puzzle, whereas our model does. Our model generates heterogeneous trading strategies—some investors are momentum traders, others are contrarians—and simultaneously produces momentum in stock returns; it does so with only one parameter, the meeting intensity λ.

6 Information Percolation and Reversal

Social interactions are natural propagators of rumors. In this section, we show that a "rumor" can generate a phase of price over-shooting followed by a phase of price correction. Under certain conditions, this convergence pattern can jointly produce short-term momentum and long-term reversal.

6.1 Introducing a Rumor

We introduce a "rumor" in the setup of Section 3. We follow Peterson and Gist (1951) and define a rumor as "*an unverified account or explanation of events circulating from person to person and pertaining to an object, event, or issue in public concern.*" To introduce this aspect in our model, we consider a small modification of the previous setup and assume that agents receive signals of the form:

$$\widetilde{z}_t^i = \widetilde{U} + \widetilde{V} + \widetilde{\epsilon}_t^i \tag{14}$$

where $\widetilde{V}$ is normally distributed with zero mean and precision ν.

The *common* noise, $\widetilde{V}$, satisfies two important properties of a rumor (as defined above): *i)* it circulates from person to person and *ii)* it is unverifiable. The first property arises as we allow agents to interact through word-of-mouth communication—private signals now contain a rumor that is circulated from one agent to the other through the information percolation mechanism described in Section 3. The second property results from the signal specification in (14): private signals only reveal the sum of the fundamental value and the rumor $(\widetilde{U} + \widetilde{V})$ on average. As a result, agents cannot distinguish fundamental information from the rumor, either using past prices or their private signals. The rumor is therefore unverifiable.

Rumors do not last forever, but eventually subside. To incorporate this aspect, we assume that each agent receives a signal at time $t = 3$ that is centered on the true value of the fundamental:

$$\widetilde{Z}_3^i = \widetilde{U} + \widetilde{\epsilon}_3^i.$$

Using this signal, agents can back out the content of the rumor (on average) at time $t = 3$ and the rumor subsides. Overall, agents are aware of the rumor, but cannot learn about its content until time $t = 3$.

In the presence of a rumor, asset prices and investor asset demands do not have a closed-form solution. Theorem 3 (the proof of which is given in Appendix A.5) describes a system of recursive equations for the equilibrium price coefficients. We solve this system of equations through an efficient numerical scheme that we describe in Appendix A.5.

Theorem 3. *In the presence of a rumor, the price $\widetilde{P}$ is informationally equivalent to*

$$\widetilde{Q}_t = \widetilde{U} + \frac{\Lambda_t}{rS\Omega_t}\widetilde{V} - \frac{1}{rS\Omega_t}\widetilde{X}_t$$

where the equilibrium coefficients Ω and Λ solve a fixed-point problem given by a system of

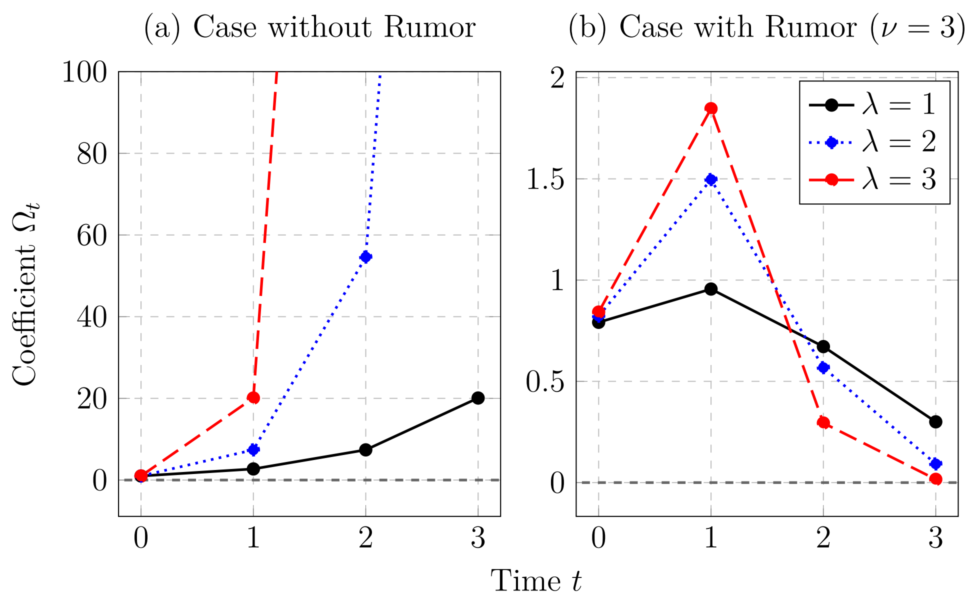

Figure 7: Price Coefficient Ω with and without Rumor

The Figure depicts the same coefficient Ω over time. Each line corresponds to a meeting intensity of $\lambda = 1$, $\lambda = 2$, and $\lambda = 3$. Panel (a) shows the coefficient Ω in the case of no rumor, while panel (b) shows the same coefficient in the case involving a rumor.

recursive equations:

$$\Omega_t = \frac{1}{rS} \sum_{j=0}^{t} \bar{\theta}_j - \sum_{j=0}^{t-1} \Omega_j \tag{15}$$

$$\Lambda_t = \sum_{j=0}^{t} \bar{\theta}_j - \sum_{j=0}^{t-1} \Lambda_j$$

in which $\bar{\theta}$ denotes the average coefficients in front of agents' private signals in their optimal demand.

The rumor has two important effects on prices, the first of which is immediately apparent when looking at the coefficient Ω. We plot this coefficient in Figure 7, both when signals contain a rumor (panel (b) with $\nu = 3$) and when they do not (panel (a)).

When signals do not contain a rumor, the coefficient Ω represents the average number of incremental signals. Panel (a) shows that this average rises as time passes by, all the more so when social interactions intensify. When signals contain a rumor (panel (b)), the coefficient Ω also keeps increasing initially, but much less so as compared to panel (a). Intuitively, agents know they possess information of lower quality due to the presence of the rumor. Therefore, they apply a discount on the actual number of signals they have—the coefficient Ω now represents a *discounted* average of incremental signals.

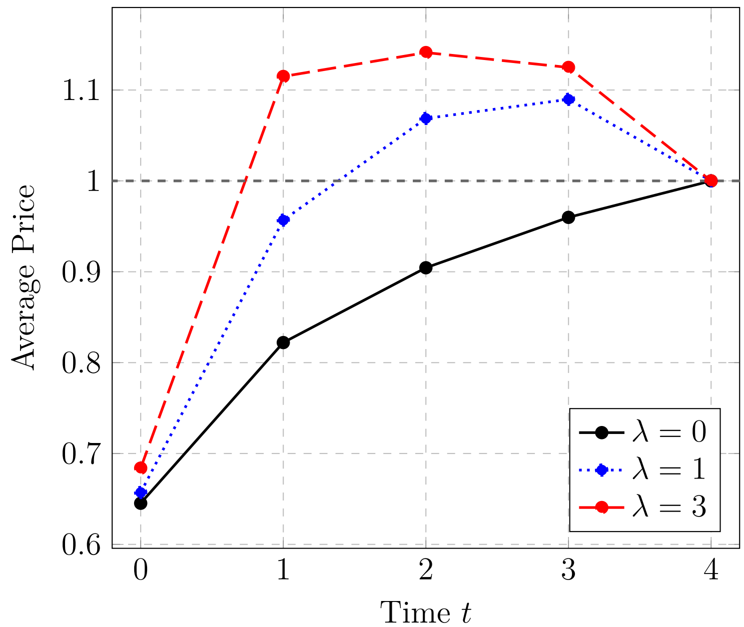

Figure 8: Average Price Convergence

The Figure depicts the average price convergence over time in the presence of a rumor. The price convergence is plotted for a meeting intensity of $\lambda = 0$ (the black solid line), $\lambda = 1$ (the blue dotted line), and $\lambda = 2$ (the red dashed line). The calibration is $H = S = \Phi = 1$, $\gamma = \frac{1}{3}$, and $\nu = 6$.

At time $t = 2$, the discounted average Ω declines: agents anticipate that they will get better information at time $t = 3$ and apply a stronger discount on their number of signals. At time $t = 3$, the discounted average number of signals reaches zero for $\lambda = 3$: when agents have collected a vast number of signals, they can accurately forecast $\widetilde{U} + \widetilde{V}$. Hence, when they get the signal that is centered on the fundamental, they basically throw away all the signals they have accumulated. Overall, the rumor induces agents to interpret their information with some caution.

A second important effect of the rumor relates to price convergence; it is precisely this effect that allows the rumor to produce long-term reversal in stock returns. We analyze price convergence in the next section.

6.2 Price Convergence and Over-Shooting

The presence of a rumor alters price convergence significantly. In particular, the rumor causes the price to "over-shoot" its fundamental value. To show this, we proceed as in Section 5.1.1: suppose we know both the fundamental value of the asset (say $\widetilde{U} = 1$) and the content of the rumor (say $\widetilde{V} = 1/3$). We can then compute how the price converges to the fundamental on average. We do so in Figure 8.

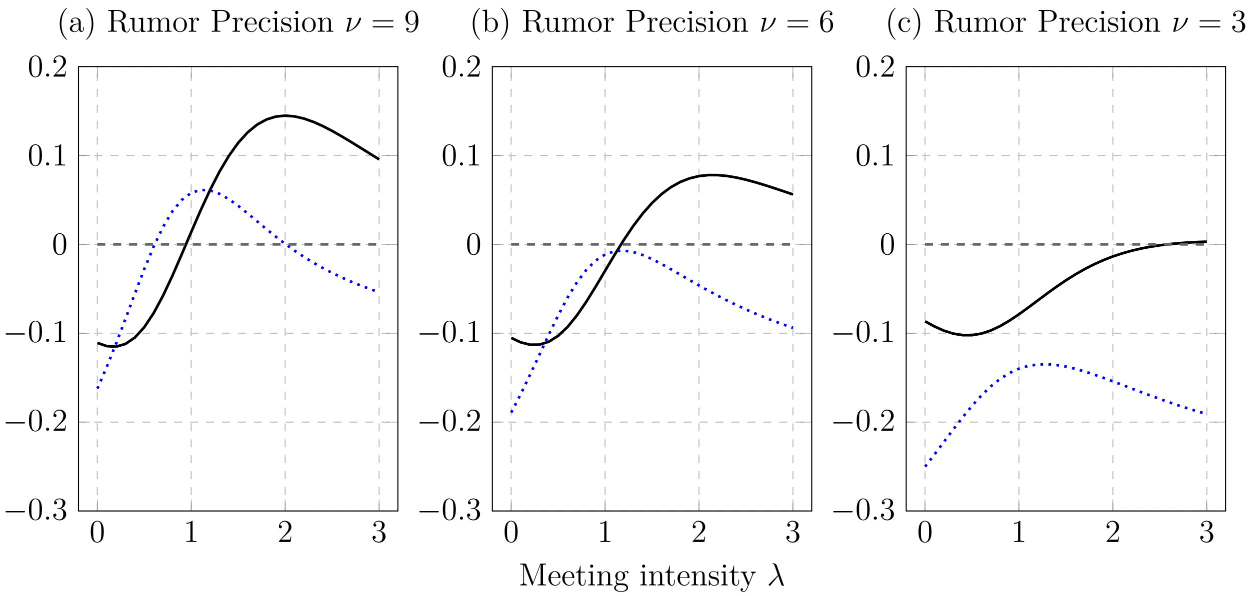

Figure 9: Serial Correlation of Return and Rumors
The Figure depicts the serial correlation of stock returns over the first- (the solid black line) and the second-period returns (the blue dotted line) as a function of the meeting intensity. Each panel corresponds to a different rumor precision ν.

Ruling out social interactions (the solid black line), prices converge slowly towards the fundamental value of the stock. This convergence pattern is similar to the one we described in Section 5.1.1. When we introduce social interactions, this pattern changes: not only do prices converge faster toward the fundamental of the stock—as in Section 5.1.1—, but, most important, they *over-shoot* the fundamental value of the stock. Social interactions and the rumor jointly produce this *hump-shaped* pattern. To see this, notice that social interactions initially accelerate price convergence towards the sum of the fundamental and the rumor causing the price to over-shoot the fundamental—the rumor diffuses through word-of-mouth communication. At time $t = 3$, agents receive information of better quality and the price corrects—the rumor subsides.

We now investigate how this convergence pattern relates to the serial correlation of stock returns.

6.3 Short-Term Momentum and Long-Term Reversal

The first phase of price "over-shooting" generates short-term momentum and the second phase of price correction generates long-term reversal. To show this, we proceed as in Section 5.1 and plot returns' serial correlation in Figure 9.

When the rumor is fairly precise (panel (a)), returns mostly exhibit momentum: despite the presence of the rumor, agents' precision rises over time. This increase in precision generates momentum, as we previously highlighted. As the precision of the rumor decreases (panel (b)), agents discount their actual number of signals more strongly. As a result, agents progressively cut back their positions—they adjust their trades to reflect that their information is of lower quality. While these portfolio adjustments do not prevent returns to exhibit momentum in the first-period, they induce reversal in the second period as the price gradually corrects. Finally, when the rumor's precision is low (panel (c)), agents become extremely cautious about their information and the improvement in their precision is not sufficient to generate momentum. As a result, returns mostly exhibit reversal.

Overall, our model can jointly generate short-term momentum—consistent with the empirical finding of Jegadeesh and Titman (1993)—and long-term reversal—consistent with the over-reaction phenomenon of De Bondt and Thaler (1985).

7 Conclusion

This paper shows, in a centralized market setting, that the diffusion of information in financial markets has a significant impact on the dynamics of asset prices. Since, in such a setting, prices do not completely reveal the fundamental, there is certainly a benefit for investors to meet and communicate with their peers.

First, we show that the percolation of information produces momentum in asset returns. Investors' precision in their estimate of the fundamental increases gradually through time, by gathering information from their peers. This increase in the conditional precision accelerates the price convergence to its path toward the fundamental, thus producing momentum in asset returns.

Information percolation is a natural way of generating heterogeneity in investments strategies. Each investor starts with one private signal and randomly meets her peers. As time shifts to the the next trading round, the diffusion of information produces heterogeneity in the number of signals that each agent gathers. Some investors optimally chose to implement momentum trading strategies, whereas others to act as contrarians. Although momentum is observable and a fraction of agents bet on it, it is not completely eliminated.

What if the information that spreads through the market is not completely accurate in the aggregate, e.g., what if it contains rumors? We add such a feature to our model, and we show that returns actually continue to exhibit momentum, yet to a lesser extent. Crucially, the momentum phase is followed by a significant reversal, in line with the widely documented under- and over-reaction patterns in stock returns.

A legitimate question is what empirical exercise would validate our model and distinguish it from other theoretical models generating the same results. We believe that natural experiments capturing an exogenous increase or decrease in the intensity of word-of-mouth communication could make a worthwhile empirical exercise. For example, Shiller (2000) relates the obvious increase in the word-of-mouth communication intensity once the telephone become effective during the 1920s with the steady increase of volatility during the same period. It would be interesting to see how the autocorrelation of asset returns evolved during the same period. Another option is to study the consequences of the Regulation Fair Disclosure, promulgated by the U.S. Securities and Exchange Commission in August 2000. This regulation forbids firms and their insiders to provide information to some investors (often large institutional investors). Hence, after August 2000 there should be less information propagated through the word-of-mouth communication channel. If our theory is consistent, there should also be less momentum.

This research shows the road for further questions. First, our setup considers a single asset. We plan to analyze how information propagates into financial market with multiple assets. We conjecture that word-of-mouth communication could generate rich dynamics of the conditional correlation among assets. Second, still in a multiple asset setting, what would be the implications if some investors talk more about some assets than others? We conjecture that word-of-mouth communication could generate informational (dis)advantages and have an impact on portfolio flows (Brennan and Cao, 1997). Finally, we intend to relax the assumption of truthful revelation of information and study conditions under which agents find it beneficial—or not—to tell the truth (Stein, 2008).

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A Appendix

A.1 Information Percolation

A.1.1 Distribution of Incremental Signals: Closed-Form Solution

To obtain the closed-form solution for the distribution π of incremental signals, we first derive the equation for its dynamics.

Lemma 1. *The probability density function π over the additional number of signals collected by each agent satisfies*

$$\frac{d}{dt}\pi_t(n) = -\lambda\pi_t(n) + \lambda(\pi_t * \mu_t)(n), \quad \pi_0 = \delta_{n=1}. \tag{16}$$

Proof. We compute the distribution of new signals that have been gathered between time 0 and time T. Denote by X_{t_i} the number of new signals gathered if a meeting occurs at time t_i and observe that it is distributed as

$$X_{t_i} \sim \mu\left(t_i, \cdot\right)$$

where the distribution $\mu\left(t, x\right)$ satisfies the differential equation in (1). Furthermore, the number $N\left(T\right)$ of meetings that took place between time 0 and T is a Poisson counter with intensity λ; accordingly, the total number Y_T of new signals gathered between time 0 and T is given by $\sum_{i=1}^{N(T)} X_{t_i}$. We now characterize its distribution. First, observe that Y_T, conditional on the set of times $\{0 \le t_1 \le t_2 \le ... \le t_{N(T)} \le T\}$ at which a meeting occurs (up to time T) and the total number of meetings $N\left(T\right)$ (that is, conditioning on the whole trajectory A_T^N of the Poisson process), is distributed as

$$\begin{aligned} Y_T|A_T^N &\sim \int_{\mathbb{R}^{N-1}} \mu\left(Y_{t_N} - Y_{t_{N-1}}, t_N\right) \mathrm{d}\mu\left(Y_{t_{N-1}} - Y_{t_{N-2}}, t_{N-1}\right) ...\mathrm{d}\mu\left(Y_{t_1} - 0, t_1\right) \\ &\equiv \int_{\mathbb{R}^{N-1}} \mu\left(X_{t_N}, t_N\right) \mathrm{d}\mu\left(X_{t_{N-1}}, t_{N-1}\right) ...\mathrm{d}\mu\left(X_{t_1}, t_1\right). \end{aligned} \tag{17}$$

Second, observe that the distribution $\mu\left(X_{t_i}, t_i\right)$ of increment can be expressed as a translation $\mathcal{T}$ of the type measure μ and the increment x. Hence, the distribution in (17) may be written as

$$Y_T|A_T^N \sim \Gamma_{i=1}^{N(T)} \mu_{t_i}$$

where, for any probability measures $\alpha_1, ..., \alpha_k$, we write $\Gamma_{i=1}^k = \alpha_1 * \alpha_2 * ... * \alpha_k$.

Now, observe that each t_i in the sequence of meetings $\{0 \le t_1 \le t_2 \le ... \le t_{N(\tau)} \le T\}$ conditional on $N\left(T\right)$ is uniformly distributed over T; accordingly, we have that

$$Y_T|N\left(T\right) \sim \Gamma_{i=1}^{N(T)} \frac{1}{T} \int_0^T \mu_{t_i} \mathrm{d}t_i = \left(\frac{1}{T^{N(T)}} \left(\int_0^T \mu_s \mathrm{d}s \right)^{*N(T)} \right)$$

where $*n$ denotes the $n-$fold convolution.

Finally, since $N\left(T\right)$ is a Poisson(λ) counter, we have

$$Y_T \sim \sum_{k=0}^{\infty} e^{-\lambda T} \frac{(\lambda T)^k}{k!} \frac{1}{T^k} \left(\int_0^T \mu_s \mathrm{d}s \right)^{*k} = \sum_{k=0}^{\infty} e^{-\lambda T} \frac{\lambda^k}{k!} \left(\int_0^T \mu_s \mathrm{d}s \right)^{*k}.$$

Using the fact that, by Taylor expansion, e^x is equivalently written as $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, we can write

$$\pi_T = e^{\lambda\left(\int_0^T \mu_s \mathrm{d}s - T\right)}.$$

Differentiating this expression, we obtain (16). ∎

A.2 Proof of Proposition 1

[TO BE COMPLETED]

A.2.1 Numerical Approach: Fourier Transform

To obtain the distribution of incremental signals numerically, we proceed through discrete Fourier transforms. Denote the Fourier transform of μ and π by $\widehat{\mu}_t(z) := \int_{\mathbb{R}} e^{izn} \mathrm{d}\mu_t(n)$ and $\widehat{\pi}_t(z) := \int_{\mathbb{R}} e^{izn} \mathrm{d}\pi_t(n)$, respectively, where $i = \sqrt{-1}$ and $z \in \mathbb{R}$. As in Duffie and Manso (2007), $\widehat{\mu}$ is given in closed form:

$$\widehat{\mu}_t(z) = \frac{\widehat{\mu}_0(z)}{e^{\lambda t}(1 - \widehat{\mu}_0(z)) + \widehat{\mu}_0(z)}$$

where $\widehat{\mu}_0(z) = e^{iz}$ since $\mu_0(n)$ is a Dirac mass at 1.

To obtain $\widehat{\pi}$, we integrate (16) and get the following equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\pi}_t(z) = -\lambda \widehat{\pi}_t(z) + \lambda \widehat{\mu}_t(z) \widehat{\pi}_t(z),$$

the solution of which is also available in closed form:

$$\widehat{\pi}_t(z) = \frac{\widehat{\pi}_0(z) e^{i\pi}}{e^{\lambda t}(\widehat{\mu}_0(z) - 1) - \widehat{\mu}_0(z)}$$

where $\widehat{\pi}_0(z) = e^{iz}$ since $\pi_0(n)$ is a Dirac mass at 1.

We can now recover π numerically by using the inverse Fourier formula. To do so, notice that both distributions $\mu(n)$ and $\pi(n)$ are so-called *lattice* distributions for which every possible realization of n can be represented as $a + bk$ where k only takes integral values—in our case, $n \in \mathbb{N}$, $a = \frac{N}{2}$, and $b = \frac{1}{2}$. For this class of discrete distributions, the inverse Fourier formula writes

$$P\left[X = x_k\right] = \frac{b}{2\pi} \int_{-\pi/b}^{\pi/b} e^{-izx_k} \widehat{\mu}_t\left(z\right) \mathrm{d}z. \tag{18}$$

To compute the integral in (18), we use fast Fourier Transform: we rewrite (18) as

$$P\left[X = x_k\right] = \frac{b}{\pi} \int_{0}^{\pi/b} e^{-izx_k} \widehat{\mu}_t\left(z\right) \mathrm{d}z.$$

We then use the trapezoid rule rule to discretize the integral:

$$P\left[X = x_k\right] \approx \frac{b}{\pi} \frac{e^{-i\frac{\pi}{b}x_k} \widehat{\mu}_t\left(\frac{\pi}{b}\right) \Delta z}{2} + \frac{b}{\pi} \sum_{j=0}^{M-1} \delta_j e^{-ij\Delta z x_k} \widehat{\mu}_t\left(j\Delta z\right) \Delta z$$

with $\delta_j = \frac{1}{2}$ if $j = 1$ and $\delta_j = 1$ otherwise. We need to choose Δz such that the upper integration bound $\frac{\pi}{b} = 2\pi$ is reached by z; accordingly, we set $\Delta z \equiv \frac{2\pi}{M}$. Furthermore, the grid points for x_k are $x_k = -d + \lambda k,\ j \in \mathbb{N}$ and the fast Fourier transform method imposes that $\lambda \Delta z = \frac{2\pi}{M}$, i.e., $\lambda = 1$. Since $x_k \in \mathbb{N}$, we must have that $d = 0$. As a result, $P\left[X = x_k\right]$ takes the form of a discrete Fourier transform:

$$P\left[X = x_k\right] \approx \frac{1}{M} \sum_{j=0}^{M-1} g_j e^{-jk\frac{2\pi}{M}i}$$

with

$$g_j = \delta_j \frac{M}{2\pi} \widehat{\mu}_t\left(j\frac{2\pi}{M}\right)\frac{2\pi}{M}.$$

Finally, since each agent gets an additional signal every period, we must use the discrete Fourier transform sequentially: to derive the distribution π at time $t+1$, we use the numerical expression for $\mu_t(n)$ that we computed at time t and compute $\widehat{\mu}_0(z) = \sum_{k=1}^{N} e^{izk} f_0(k)$ where

$$f_0(n) = \begin{cases} \mu_t(n-1) & \text{if } n \geq t+1 \\ 0 & \text{otherwise} \end{cases}.$$

We then substitute $\widehat{\mu}_0(n)$ into $\widehat{\pi}_t(n)|_{t=1}$ and apply the inverse Fourier formula, which yields π_{t+1}.

A.3 Proof of Theorem 1

We provide the proof for a two trading session economy, that is, we eliminate for ease of exposition dates $t = 2$ and $t = 3$. Once the equilibrium quantities are written in a recursive form, as in Brennan and Cao (1997), or in He and Wang (1995), it is straightforward to derive the full recursive equilibrium solution.

The model is solved backwards, starting from date 1 and then going back to date 0. First, conjecture that prices in period 0 and period 1 are

$$\widetilde{P}_0 = \beta_0 \widetilde{U} - \alpha_{0,0} \widetilde{X}_0 \tag{19}$$

$$\widetilde{P}_1 = \beta_1 \widetilde{U} - \alpha_{1,0} \widetilde{X}_0 - \alpha_{1,1} \widetilde{X}_1 \tag{20}$$

Consider the normalized price signal in period zero (which is informationally equivalent to $\widetilde{P}_0$):

$$\widetilde{Q}_0 = \frac{1}{\beta_0} \widetilde{P}_0 = \widetilde{U} - \frac{\alpha_{0,0}}{\beta_0} \widetilde{X}_0 \tag{21}$$

Replace $\widetilde{X}_0$ from (21) into (20) to obtain

$$\widetilde{P}_1 = \varphi_1 \widetilde{U} + \xi_1 \widetilde{Q}_0 - \alpha_{1,1} \widetilde{X}_1 \tag{22}$$

where $\varphi_1 = \beta_1 - \alpha_{1,0} \frac{\beta_0}{\alpha_{0,0}}$ and $\xi_1 = \alpha_{1,0} \frac{\beta_0}{\alpha_{0,0}}$. These coefficients are to be determined in equilibrium. We normalize the price signal in period $t = 1$ and obtain $\widetilde{Q}_1$:

$$\widetilde{Q}_1 = \frac{1}{\varphi_1} \left(\widetilde{P}_1 - \xi_1 \widetilde{Q}_0 \right) = \widetilde{U} - \frac{\alpha_{1,1}}{\varphi_1} \widetilde{X}_1$$

Observing $\{\widetilde{Q}_0, \widetilde{Q}_1\}$ is equivalent with observing $\{\widetilde{P}_0, \widetilde{P}_1\}$. We conjecture the following relationships (see Admati 1985):

$$\frac{\alpha_{0,0}}{\beta_0} = \frac{1}{\gamma S \Omega_0} \tag{23}$$

$$\frac{\alpha_{1,1}}{\varphi_1} = \frac{1}{\gamma S \Omega_1} \tag{24}$$

where Ω_t is the cross-sectional average of the number of additional signals at time $t = 0, 1$, $\Omega_t = \sum_{\omega \in \mathbb{N}} \omega \pi_t(\omega)$. In our setup, $\Omega_0 = 1 \; \forall \lambda$, $\Omega_1 = 1$ if $\lambda = 0$, and $\Omega_1 > 1$ if $\lambda > 0$. Relationships (23) and (24), which make the calculations that follow straightforward, are to be verified once the solution is obtained. Thus, the normalized price signals, informationally equivalent with prices, are:

$$\widetilde{Q}_0 = \widetilde{U} - \frac{1}{\gamma S \Omega_0} \widetilde{X}_0 \tag{25}$$

$$\widetilde{Q}_1 = \widetilde{U} - \frac{1}{\gamma S \Omega_1} \widetilde{X}_1 \tag{26}$$

A.3.1 Period 1

Consider an investor i who, at date $t=1$, collects $\omega_1^i \geq 1$ additional signals. At date $t=1$, investor i chooses $\widetilde{D}_1^i$ to maximize expected utility of final wealth:

$$\max_{\widetilde{D}_1^i} \mathbb{E}\left[-e^{-\frac{1}{\gamma}\widetilde{W}_2^i}\middle|\widetilde{F}_1^i\right]$$

where the final wealth at date $t=2$ (at liquidation) is

$$\widetilde{W}_2^i = X^i \widetilde{P}_0 + \widetilde{D}_0^i\left(\widetilde{P}_1 - \widetilde{P}_0\right) + \widetilde{D}_1^i\left(\widetilde{U} - \widetilde{P}_1\right) \tag{27}$$

and $\widetilde{F}_1^i$ represents the total information available at date $t=1$. This information is given by $\widetilde{Z}_1^i$, $\widetilde{Z}_0^i$ (private signals) and $\widetilde{Q}_1$, $\widetilde{Q}_0$ (public signals, informationally equivalent with prices and defined in (25) and (26)). Note that $\widetilde{Z}_0^i$ represent only one signal of precision S, but $\widetilde{Z}_1^i$ represent the average of the ω_1^i additional signals collected by the investor at date $t=1$ (ω_1^i signals of equal precision S are informationally equivalent with a signal equal to their average and having precision $\omega_1^i S$). With this information at hand at date $t=1$, investor i will try to forecast $\widetilde{U}$. The state variables corresponding to investor i are therefore

Precision →	$\frac{1}{H}$	$\frac{1}{\omega_1^i S}$	$\frac{1}{S}$	$\frac{1}{\Phi}$	$\frac{1}{\Phi}$	
Variable ↓	$\widetilde{U}$	$\widetilde{\varepsilon}_1^i$	$\widetilde{\varepsilon}_0^i$	$\widetilde{X}_1$	$\widetilde{X}_0$	$\mathbb{E}[\cdot]$
$\widetilde{U}$	1	0	0	0	0	0
$\widetilde{Z}_1^i$	1	1	0	0	0	0
$\widetilde{Z}_0^i$	1	0	1	0	0	0
$\widetilde{Q}_1$	1	0	0	$-\frac{1}{\gamma S \Omega_1}$	0	0
$\widetilde{Q}_0$	1	0	0	0	$-\frac{1}{\gamma S \Omega_0}$	0

It is straightforward to calculate

$$K_1^i = Var^{-1}\left[\widetilde{U}\middle|\widetilde{Z}_1^i, \widetilde{Z}_0^i, \widetilde{Q}_1, \widetilde{Q}_0\right]$$
$$\widetilde{\mu}_1^i = \mathbb{E}\left[\widetilde{U}\middle|\widetilde{Z}_1^i, \widetilde{Z}_0^i, \widetilde{Q}_1, \widetilde{Q}_0\right]$$

by using the projection theorem:

Theorem 4 (Projection Theorem). *Consider a n-dimensional normal random variable*

$$(\theta, s) \sim N\left(\begin{bmatrix}\mu_\theta \\ \mu_s\end{bmatrix}, \begin{bmatrix}\Sigma_{\theta,\theta} & \Sigma_{\theta,s} \\ \Sigma_{s,\theta} & \Sigma_{s,s}\end{bmatrix}\right)$$

The conditional density of θ given s is normal with conditional mean

$$\mu_\theta + \Sigma_{\theta,s}\Sigma_{s,s}^{-1}(s - \mu_s)$$

and variance-covariance matrix

$$\Sigma_{\theta,\theta} - \Sigma_{\theta,s}\Sigma_{s,s}^{-1}\Sigma_{s,\theta}$$

provided $\Sigma_{s,s}$ is non-singular.

We get

$$
\begin{aligned}
K_1^i &= H + S\left(1 + \omega_1^i\right) + \gamma^2 S^2 \Phi\left(\Omega_0^2 + \Omega_1^2\right) \\
\widetilde{\mu}_1^i &= \frac{1}{K_1^i}\left[S\widetilde{Z}_0^i + S\omega_1^i \widetilde{Z}_1^i + \gamma^2 S^2 \Phi\left(\Omega_0^2 \widetilde{Q}_0 + \Omega_1^2 \widetilde{Q}_1\right)\right]
\end{aligned} \tag{28}
$$

The optimal demand of trader i in period 1 has a standard form (from the normality of distribution assumption in conjunction with the exponential utility function):

$$
\widetilde{D}_1^i = \gamma K_1^i\left(\widetilde{\mu}_1^i - \widetilde{P}_1\right) \tag{29}
$$

Replace (28) in (29) to obtain

$$
\widetilde{D}_1^i = \gamma\left[S\widetilde{Z}_0^i + S\omega_1^i \widetilde{Z}_1^i + \gamma^2 S^2 \Phi\left(\Omega_0^2 \widetilde{Q}_0 + \Omega_1^2 \widetilde{Q}_1\right) - K_1^i \widetilde{P}_1\right] \tag{30}
$$

We can now integrate the optimal demands to get the total demand. We follow the convention used by Admati (1985) that implies $\int_0^1 \widetilde{Z}_j^i = \widetilde{U}$, a.s.. More important, we have now to keep track of the heterogeneity in information endowments when aggregating all individual demands. In particular, at time $t = 1$ there is an infinity of types of investors with respect to their number of signals, and in each such type there is a continuum of investors. Consequently, the total demand at time $t = 1$ is

$$
\widetilde{D}_1 = \int_0^1 \widetilde{D}_1^i = \sum_{\omega_1^i = 1}^{\infty}\left[\pi_1(\omega_1^i)\int_{\omega_1^i} \widetilde{D}_1^i\right]
$$

which yields

$$
\widetilde{D}_1 = \gamma\left[S\left(\Omega_0 + \Omega_1\right)\widetilde{U} + \gamma^2 S^2 \Phi\left(\Omega_0^2 \widetilde{Q}_0 + \Omega_1^2 \widetilde{Q}_1\right) - K_1 \widetilde{P}_1\right] \tag{31}
$$

where K_1 is the average precision across the entire population of agents:

$$
K_1 \equiv \sum_{\omega_1^i = 1}^{\infty} K_1^i(\omega_1^i)\pi_1(\omega_1^i) = H + S\left(\Omega_0 + \Omega_1\right) + \gamma^2 S^2 \Phi\left(\Omega_0^2 + \Omega_1^2\right)
$$

Replace (26) in (31) to obtain

$$
\widetilde{D}_1 = \gamma\left[\left(S\Omega_0 + S\Omega_1 + \gamma^2 S^2 \Phi \Omega_1^2\right)\widetilde{U} + \gamma^2 S^2 \Phi \Omega_0^2 \widetilde{Q}_0 - \gamma \Phi S \Omega_1 \widetilde{X}_1 - K_1 \widetilde{P}_1\right]
$$

The market clearing condition is $\widetilde{D}_1 = \widetilde{X}_0 + \widetilde{X}_1$. Once we impose market clearing, we can use the conjectured $\widetilde{P}_1$ equation (22) to get the undetermined coefficients φ_1, ξ_1, and $\alpha_{1,1}$:

$$
\begin{aligned}
\varphi_1 &= \frac{S\Omega_1\left(1 + \gamma^2 S \Phi \Omega_1\right)}{K_1}, \\
\xi_1 &= \frac{S\Omega_0\left(1 + \gamma^2 S \Phi \Omega_0\right)}{K_1} \\
\alpha_{1,1} &= \frac{1 + \gamma^2 S \Phi \Omega_1}{\gamma K_1}
\end{aligned}
$$

From these solutions, we can verify that, indeed, $\frac{\alpha_{1,1}}{\varphi_1} = \frac{1}{\gamma S \Omega_1}$. Hence, (24) is now verified. The undetermined coefficients of $\widetilde{P}_1$ from the conjectured form (20) are

$$\beta_1 = \frac{K_1 - H}{K_1}$$
$$\alpha_{1,0} = \frac{1 + \gamma^2 S \Phi \Omega_0}{\gamma K_1}$$
$$\alpha_{1,1} = \frac{1 + \gamma^2 S \Phi \Omega_1}{\gamma K_1}$$

and thus

$$\widetilde{P}_1 = \frac{K_1 - H}{K_1} \widetilde{U} - \frac{1 + \gamma^2 S \Phi \Omega_0}{\gamma K_1} \widetilde{X}_0 - \frac{1 + \gamma^2 S \Phi \Omega_1}{\gamma K_1} \widetilde{X}_1 \tag{32}$$

which can also be written as

$$\widetilde{P}_1 = \frac{S\Omega_0 + \gamma^2 S^2 \Phi \Omega_0^2}{\gamma K_1} \widetilde{Q}_0 + \frac{S\Omega_1 + \gamma^2 S^2 \Phi \Omega_1^2}{\gamma K_1} \widetilde{Q}_1 \tag{33}$$

Furthermore, replacing $\widetilde{P}_1$ written under the form (33) in the optimal demand written under the form (30) gives the following result:

$$\widetilde{D}_1^i = \gamma \left[S\Omega_0 \left(\widetilde{Z}_0^i - \widetilde{Q}_0 \right) + S\omega_1^i \widetilde{Z}_1^i - S\Omega_1 \widetilde{Q}_1 - \left(K_1^i - K_1 \right) \widetilde{P}_1 \right] \tag{34}$$

The coefficient of $\widetilde{P}_1$ in (34) represents the cumulative informational (dis)advantage of investor i, which we denote thereafter by $A_1^i \equiv K_1^i - K_1$. Thus, the optimal demand of investor i at time $t = 1$ is

$$\widetilde{D}_1^i = \gamma \left(S\omega_0^i \widetilde{Z}_0^i - S\Omega_0 \widetilde{Q}_0 + S\omega_1^i \widetilde{Z}_1^i - S\Omega_1 \widetilde{Q}_1 - A_1^i \widetilde{P}_1 \right) \tag{35}$$

where $\omega_0^i = \Omega_0 = 1$, but we have included them here to highlight the recursive form of the optimal demand.

A.3.2 Period 0

The problem of investor i at time $t = 0$ is

$$\max_{\widetilde{D}_0^i} \mathbb{E} \left[-e^{-\frac{1}{\gamma} \widetilde{W}_2^i} \middle| \widetilde{Z}_0^i, \widetilde{Q}_0 \right]$$

where the final wealth is given in (27). Observe that, at time $t = 0$, investor i needs to estimate $\widetilde{U}$, $\widetilde{P}_1$ and $\widetilde{D}_1^i$, after observing $\widetilde{Z}_0^i$ and $\widetilde{Q}_0$. $\widetilde{P}_1$ and $\widetilde{D}_1^i$ are given by (32) and (34). The maximization problem then becomes:

$$\max_{\widetilde{D}_0^i} \mathbb{E} \left[-e^{-\frac{1}{\gamma} \left[X^i \widetilde{P}_0 + \widetilde{D}_0^i \left(\widetilde{P}_1 - \widetilde{P}_0 \right) + \widetilde{D}_1^i \left(\widetilde{U} - \widetilde{P}_1 \right) \right]} \middle| \widetilde{Z}_0^i, \widetilde{Q}_0 \right]$$

Lemma 2. *When an agent builds her portfolio, her future number of signals is irrelevant.*

Proof. We prove the claim in a slightly more general context: for the proof only, suppose agent i

starts with an arbitrary number n_0 of signals at time $t=0$. The value function V^i of agent i is then given by

$$\begin{aligned} V^i(n_0,W_0) &= e^{-\frac{1}{\gamma}W_0} \max_{\widetilde{D}_0^i(n_0)} \mathbb{E}\left[-e^{-\frac{1}{\gamma}\left(\widetilde{D}_0^i(n_0)\left(\widetilde{P}_1-\widetilde{P}_0\right)+\widetilde{D}_1^i(n_1)\left(\widetilde{U}-\widetilde{P}_1\right)\right)} \middle| \widetilde{Z}_0^i, \widetilde{Q}_0 \right] \\ &= e^{-\frac{1}{\gamma}W_0} \max_{\widetilde{D}_0^i(n_0)} \sum_{k=1}^{\infty} \mu_1(k) \underbrace{\mathbb{E}\left[-e^{-\frac{1}{\gamma}\left(\widetilde{D}_0^i(n_0)\left(\widetilde{P}_1-\widetilde{P}_0\right)+\widetilde{D}_1^i(k)\left(\widetilde{U}-\widetilde{P}_1\right)\right)} \middle| \widetilde{Z}_0^i, \widetilde{Q}_0; n_1=k \right]}_{g\left(n_0,k,\widetilde{D}_0^i\right)}. \end{aligned}$$

The function g represents an expectation of an exponential affine quadratic normal variable. To derive its explicit form, we use the following theorem.

Theorem 5. *Consider a random vector $z \sim N(0,\Sigma)$. Then,*

$$E\left[e^{z'Fz+G'z+H}\right] = |I-2\Sigma F|^{-\frac{1}{2}}\, e^{\frac{1}{2}G'(I-2\Sigma F)^{-1}\Sigma G+H}.$$

Tedious computations then show that

$$g\left(n_0,k,\widetilde{D}_0^i\right) = -\left|I-2\Sigma(n_0,k)F(k)\right|^{-\frac{1}{2}}\, e^{\frac{1}{2}G\left(n_0,k,\widetilde{D}_0^i\right)'(I-2\Sigma(n_0,k)F(k))^{-1}\Sigma(n_0,k)G\left(n_0,\widetilde{D}_0^i\right)+H\left(n_0,k,\widetilde{D}_0^i\right)} \tag{36}$$

where

$$\Sigma(n_0,k) = \begin{pmatrix} \frac{1}{K_0^i(n_0)} & \frac{K_1-H}{K_1K_0^i(n_0)} & \frac{1}{K_0^i(n_0)}-\frac{1}{K_1^i(k)} \\ \frac{K_1-H}{K_1K_0^i(n_0)} & \frac{(K_1^i(k)-S(k-n_0))(1+\gamma^2S\Phi\Omega_{t+1})^2}{K_1^2K_0^i(n_0)\gamma^2\Phi} & \frac{K_1-H}{K_1K_0^i(n_0)} \\ \frac{1}{K_0^i(n_0)}-\frac{1}{K_1^i(k)} & \frac{K_1-H}{K_1K_0^i(n_0)} & \frac{1}{K_0^i(n_0)}-\frac{1}{K_1^i(k)} \end{pmatrix},$$

$$F(k) = \begin{pmatrix} 0 & \frac{1}{2}K^i(k) & -\frac{1}{2}K^i(k) \\ \frac{1}{2}K^i(k) & -K^i(k) & \frac{1}{2}K^i(k) \\ -\frac{1}{2}K^i(k) & \frac{1}{2}K^i(k) & 0 \end{pmatrix},$$

$$G\left(n_0,k,\widetilde{D}_0^i\right) = \begin{pmatrix} \frac{K_1^i(k)S(\Omega_0(H+n_0S(1+\gamma^2S\Omega_0\Phi))\widetilde{Q}_0-K_0n_0\widetilde{Z}_0^i)}{K_1K_0^i(n_0)} \\ -\frac{\widetilde{D}_0^i}{\gamma}-\frac{2K_1^i(k)S(\Omega_0(H+n_0S(1+\gamma^2S\Phi\Omega_0))\widetilde{Q}_0-K_0n_0\widetilde{Z}_0^i)}{K_1K_0^i(n_0)} \\ \frac{K_1^i(k)S(\Omega_0(H+n_0S(1+\gamma^2S\Omega_0\Phi))\widetilde{Q}_0-K_0n_0\widetilde{Z}_0^i)}{K_1K_0^i(n_0)} \end{pmatrix},$$

$$H\left(n_0,k,\widetilde{D}_0^i\right) = -\frac{\widetilde{P}_0}{\gamma}\widetilde{D}_0^i - \frac{S\begin{pmatrix} K_1^i(k)\gamma S((K_0n_0+H(\Omega_0-n_0))\widetilde{Q}_0-K_0n_0\widetilde{Z}_0^i)^2 \\ +K_1K_0^i(n_0)\widetilde{D}_0^i\begin{pmatrix} (K_0n_0+K_1\gamma^2S\Phi\Omega_0^2+H(\Omega_0-n_0))\widetilde{Q}_0 \\ +n_0S\Omega_1(1+\gamma^2S\Phi\Omega_1)\widetilde{Z}_0^i \end{pmatrix} \end{pmatrix}}{\gamma K_1^2K_0^i(n_0)^2}.$$

Further computations show that

$$h(n_0,k) \equiv |I-2\Sigma(n_0,k)F(k)| = \frac{K_1^i(k)}{K_1^2K_0^i(n_0)\gamma^2\Phi}\begin{pmatrix} H^2\gamma^2\Phi+H(1+\gamma^2\Phi(K_0+K_1-2H+S\Omega_1)) \\ +S\begin{pmatrix} n_0(1+\gamma^2S\Phi\Omega_1)^2 \\ +\gamma^2S\Phi\Omega_0^2(2+\gamma^2\Phi(K_1-H+S(\Omega_0+\Omega_1))) \end{pmatrix} \end{pmatrix}$$

and

$$q\left(n_0,\widetilde{D}_0^i\right) \equiv \frac{1}{2}G\left(n_0,k,\widetilde{D}_0^i\right)'\left(I-2\Sigma(n_0,k)F(k)\right)^{-1}\Sigma(n_0,k)G\left(n_0,\widetilde{D}_0^i\right)+H\left(n_0,k,\widetilde{D}_0^i\right)$$

$$= \frac{1}{2K_0^i(n_0)\gamma^2\left(H^2\gamma^2\Phi + HX + S\left(n_0 Z_1^2 + \gamma^2 S\Phi Y \Omega_0^2\right)\right)}$$

$$\times \left(\begin{array}{c} 2K_0^i(n_0)\gamma \widetilde{D}_0^i \left(\begin{array}{c} \widetilde{P}_0\left(H^2\gamma^2\Phi + HX + S\left(n_0 Z_1^2 + \gamma^2 S\Phi Y\Omega_0^2\right)\right) \\ -S\left(\gamma^2\Phi\Omega_0\widetilde{Q}_0\left(H + S\Omega_0\left(\gamma^2\Phi(K_1 + S(\Omega_0+\Omega_1)) + 2\right)\right) + n_0 Z_1^2 \widetilde{Z}_0^i\right) \end{array} \right) \\ +K_0^i(n_0) Z_1^2 (\widetilde{D}_0^i)^2 - \gamma^4\Phi\left(S\Omega_0\widetilde{Q}_0(H + n_0 S Z_0) - K_0 n_0 S\widetilde{Z}_0^i\right)^2 \end{array} \right)$$

with

$$\begin{aligned} X &= 1 + \gamma^2\Phi(K_0 + K_1 - 2H + S\Omega_1), \\ Y &= 2 + \gamma^2\Phi(K_1 - H + S(\Omega_0 + \Omega_1)), \\ Z_t &= 1 + \gamma^2 S\Phi\Omega_t. \end{aligned}$$

Plugging these expressions into (36), agent i solves

$$V^i(n_0, W_0) = e^{-\frac{1}{\gamma}W_0} \underbrace{\left(\sum_{k=1}^{\infty}\mu_1(k)h(n_0,k)\right)}_{\text{anticipation of future signals}} \max_{\widetilde{D}_0^i(n_0)} -e^{q\left(n_0,\widetilde{D}_0^i\right)} \tag{37}$$

and it follows that her portfolio decision is independent of her expectation regarding her future number of signals. ■

To obtain agent i's optimal demand, we solve the problem in (37) and impose the first-order condition

$$\frac{\partial}{\partial \widetilde{D}_0^i} q\left(n_0, \widetilde{D}_0^i\right) = 0.$$

We integrate the resulting optimal demand and impose market clearing in order to solve for the undetermined coefficients of $\widetilde{P}_0$, i.e., β_0 and $\alpha_{0,0}$. The solutions for these coefficients are:

$$\begin{aligned} \beta_0 &= \frac{K_0 - H}{K_0} \\ \alpha_{0,0} &= \frac{1 + \gamma^2 S\Phi\Omega_0}{\gamma K_0} \end{aligned} \tag{38}$$

where

$$\begin{aligned} K_0 = K_0^i &= H + S + \gamma^2 S^2 \Phi\Omega_0^2 \\ \widetilde{\mu}_0^i &= \frac{1}{K_0}\left(S\widetilde{Z}_0^i + \gamma^2 S^2\Phi\Omega_0^2\widetilde{Q}_0\right) \end{aligned} \tag{39}$$

We have $K_0 = K_0^i$ in (39) because investors start with homogeneous information endowments at time 0, i.e., all have one signal. In other words, the cumulative informational (dis)advantage is zero for all investors: $A_0^i = 0$, $\forall i$. The optimal demand of investor i at time $t = 0$ is

$$\widetilde{D}_0^i = \gamma\left(S\omega_0^i\widetilde{Z}_0^i - S\Omega_0\widetilde{Q}_0 - A_0^i\widetilde{P}_0\right) \tag{40}$$

where we have added the last term to show that the demand at time $t = 0$ takes a similar form with the demand at time $t = 1$, expressed in equation (35). At this point, we can use (38) and (39) to verify that, indeed, $\frac{\alpha_{0,0}}{\beta_0} = \frac{1}{\gamma S\Omega_0}$. Hence, (23) is now verified. The solution can then be written in a recursive form and extended to more than 2 trading periods, as done in Theorem 1.

A.3.3 Optimal Trading Strategy

The optimal trading strategy of investor i at time $t = 1$ follows from (35) and (40):

$$\begin{aligned}\Delta\widetilde{D}_1^i \equiv \widetilde{D}_1^i - \widetilde{D}_0^i &= \gamma\left(S\omega_1^i\widetilde{Z}_1^i - S\Omega_1\widetilde{Q}_1 - A_1^i\widetilde{P}_1 + A_0^i\widetilde{P}_0\right)\\ &= \gamma\left[S\omega_1^i\widetilde{Z}_1^i - S\Omega_1\widetilde{Q}_1 - A_1^i\left(\widetilde{P}_1 - \widetilde{P}_0\right) - \left(A_1^i - A_0^i\right)\widetilde{P}_0\right]\end{aligned}$$

The cumulative informational (dis)advantage at time $t = 1$, A_1^i, has a recursive form:

$$A_1^i = A_0^i + S\omega_1^i - S\Omega_1 = A_0^i + a_1^i$$

where $a_1^i \equiv S\omega_1^i - S\Omega_1$ is the marginal informational (dis)advantage of trader i at time $t = 1$.

Thus, the optimal trading strategy becomes:

$$\begin{aligned}\Delta\widetilde{D}_1^i &= \gamma\left[S\omega_1^i\left(\widetilde{Z}_1^i - \widetilde{P}_0\right) - S\Omega_1\left(\widetilde{Q}_1 - \widetilde{P}_0\right) - A_1^i\left(\widetilde{P}_1 - \widetilde{P}_0\right)\right]\\ &= \gamma\left[\left(S\omega_1^i - S\Omega_1\right)\left(\widetilde{Z}_1^i - \widetilde{P}_0\right) + S\Omega_1\left(\widetilde{Z}_1^i - \widetilde{P}_0\right) - S\Omega_1\left(\widetilde{Q}_1 - \widetilde{P}_0\right) - A_1^i\left(\widetilde{P}_1 - \widetilde{P}_0\right)\right] \quad (41)\end{aligned}$$

Furthermore, it can be verified that

$$\begin{aligned}\widetilde{Q}_0 &= \frac{K_0\widetilde{P}_0}{S\Omega_0 + \gamma^2 S^2\Phi\Omega_0^2}\\ \widetilde{Q}_1 &= \frac{K_1\widetilde{P}_1 - K_0\widetilde{P}_0}{S\Omega_1 + \gamma^2 S^2\Phi\Omega_1^2}\end{aligned}$$

Replacing $\widetilde{Q}_1$ in (41) yields

$$\Delta\widetilde{D}_1^i = \gamma\left[a_1^i\left(\widetilde{Z}_1^i - \widetilde{P}_0\right) + S\Omega_1\left(\widetilde{Z}_1^i - \widetilde{P}_0\right) - \frac{K_1}{1 + \gamma^2 S\Phi\Omega_1}\left(\widetilde{P}_1 - \widetilde{P}_0\right) - A_1^i\left(\widetilde{P}_1 - \widetilde{P}_0\right)\right]$$

This is the same recursive form as in (10) and thus the proof of Theorem 1 is complete. □

A.4 Formula for ρ_t and Proof of Proposition 3

$$\rho_t = \frac{\left(\frac{1}{K_t} - \frac{1}{K_{t+1}}\right)\left(\left(\frac{1}{K_{t-1}} - \frac{1}{K_t}\right)\left(H + \sum_{j=0}^{t-1}(1+\gamma^2 S\Omega_j\Phi)^2\right) - \frac{1}{\gamma^2 K_t \Phi}(1+\gamma^2 S\Omega_t\Phi)^2\right)}{\left(\frac{1}{K_{t-1}} - \frac{1}{K_t}\right)^2\left(H + \frac{1}{\gamma^2\Phi}\sum_{j=0}^{t-1}(1+\gamma^2 S\Omega_j\Phi)^2\right) + \frac{1}{\gamma^2 K_t^2 \Phi}(1+\gamma^2 S\Omega_t\Phi)^2}. \tag{42}$$

Proof of Proposition 3

Shutting down social interactions, prices are obtained as a special case of (3) when the average incremental number of signal satisfies $\Omega_t \equiv 1$; accordingly, the numerator in (42) simplifies to

$$-\frac{HS\left(\gamma^2 S\Phi + 1\right)^2}{\gamma^2\Phi\left(H + St\left(\gamma^2 S\Phi + 1\right)\right)\left(H + S(t+1)\left(\gamma^2 S\Phi + 1\right)\right)^2\left(H + S(t+2)\left(\gamma^2 S\Phi + 1\right)\right)} < 0.$$

This expression is negative for any time t.

A.5 Appendix for Section 6

To prove Theorem 3, we adapt the expression for the price $\widetilde{P}$ in Brennan and Cao (1997) and write

$$\widetilde{P}_t = \beta_t \widetilde{U} + \alpha_t \widetilde{V} + \sum_{j=0}^{t-1} \xi_{j,t} \widetilde{Q}_j - \gamma_t \widetilde{X}_t.$$

The price is informationally equivalent to

$$\widetilde{Q}_t = \frac{1}{\beta_t} \widetilde{P}_t - \sum_{j=0}^{t-1} \xi_{j,t} \widetilde{Q}_j = \widetilde{U} + \frac{\alpha_t}{\beta_t} \widetilde{V} - \frac{\gamma_t}{\beta_t} \widetilde{X}_t.$$

Furthermore, we can write agent i's individual demand as

$$\widetilde{D}_t^i = \omega_t^i \widetilde{P}_t + \sum_{k=0}^{t} \lambda_k^i \widetilde{Q}_k + \sum_{k=0}^{t} \theta_k^i \widetilde{Z}_k^i.$$

By the law of large numbers, we have that $\int_{i\in[0,1]} \widetilde{Z}_k^i \mathrm{d}\mu(i) = \widetilde{U} + \widetilde{V}$. As a result, when we aggregate individual demands, we obtain

$$\int_{i\in[0,1]} \widetilde{D}_t^i \mathrm{d}\mu(i) = \bar{\omega}_t \widetilde{P}_t + \sum_{k=0}^{t} \bar{\lambda}_k \widetilde{Q}_k + \sum_{k=0}^{t} \bar{\theta}_k \widetilde{U} + \sum_{k=0}^{t} \bar{\theta}_k \widetilde{V}$$

where $\bar{\omega}_t = \sum_{\mathbb{N}} \pi_t(k) \omega_t^i(k)$, $\bar{\lambda}_t = \sum_{\mathbb{N}} \pi_t(k) \lambda_t^i(k)$, and $\bar{\theta}_t = \sum_{\mathbb{N}} \pi_t(k) \theta_t^i(k)$.

Imposing market clearing, we have

$$\sum_{k=0}^{t} \widetilde{X}_k - \sum_{k=0}^{t} \bar{\theta}_k \widetilde{U} - \sum_{k=0}^{t} \bar{\theta}_k \widetilde{V} = \bar{\omega}_t \widetilde{P}_t + \sum_{k=0}^{t} \bar{\lambda}_k \widetilde{Q}_k.$$

Substituting

$$\widetilde{P}_t = \beta_t \widetilde{Q}_t + \sum_{j=0}^{t-1} \xi_{j,t} \widetilde{Q}_j$$

into the above equation, we obtain

$$\sum_{k=0}^{t} \widetilde{X}_k - \sum_{k=0}^{t} \bar{\theta}_k \widetilde{U} - \sum_{k=0}^{t} \bar{\theta}_k \widetilde{V} = \bar{\omega}_t (\beta_t \widetilde{Q}_t + \sum_{j=0}^{t-1} \xi_{j,t} \widetilde{Q}_j) + \sum_{k=0}^{t} \bar{\lambda}_k \widetilde{Q}_k.$$

Furthermore, notice that

$$\widetilde{X}_k = \frac{\beta_k}{\gamma_k} \left(\widetilde{U} + \frac{\alpha_k}{\beta_k} \widetilde{V} - \widetilde{Q}_k \right).$$

Substituting and regrouping, we obtain

$$\widetilde{X}_t + \sum_{k=0}^{t-1} \frac{\beta_k}{\gamma_k} \left(\widetilde{U} + \frac{\alpha_k}{\beta_k} \widetilde{V} - \widetilde{Q}_k \right) - \sum_{k=0}^{t} \bar{\theta}_k \widetilde{U} - \sum_{k=0}^{t} \bar{\theta}_k \widetilde{V} = (\bar{\omega}_t \beta_t + \bar{\lambda}_t) \widetilde{Q}_t + \sum_{j=0}^{t-1} (\xi_{j,t} \bar{\omega}_t + \bar{\lambda}_j \widetilde{Q}_j),$$

or, equivalently,

$$\widetilde{X}_t + \left(\sum_{k=0}^{t-1} \frac{\beta_k}{\gamma_k} - \bar{\theta}_k\right) \widetilde{U} + \left(\sum_{k=0}^{t-1} \frac{\beta_k}{\gamma_k}\frac{\alpha_k}{\beta_k} - \sum_{k=0}^{t} \bar{\theta}_k\right) \widetilde{V} - \sum_{k=0}^{t-1} \frac{\beta_k}{\gamma_k} \widetilde{Q}_k \equiv f(\{\widetilde{Q}_j\}_{j=0}^t).$$

The right-hand side of this equation is only a function of $\{\widetilde{Q}_j\}_{j=0}^t$. By separation of variables, the left-hand side must also be a function of $\{\widetilde{Q}_j\}_{j=0}^t$ only. Hence, it must be that

$$-\left(\sum_{k=0}^{t-1} \frac{\beta_k}{\gamma_k} - \bar{\theta}_k\right) \left(\widetilde{U} + \frac{\sum_{k=0}^{t-1} \frac{\beta_k}{\gamma_k}\frac{\alpha_k}{\beta_k} - \sum_{k=0}^{t} \bar{\theta}_k}{\sum_{k=0}^{t-1} \frac{\beta_k}{\gamma_k} - \bar{\theta}_k^1} \widetilde{V} - \frac{1}{\sum_{k=0}^{t-1} \frac{\beta_k}{\gamma_k} - \bar{\theta}_k} \widetilde{X}_t\right) \equiv \widetilde{Q}_t.$$

This equality holds if and only if

$$\frac{\sum_{k=0}^{t-1} \frac{\beta_k}{\gamma_k}\frac{\alpha_k}{\beta_k} - \sum_{k=0}^{t} \bar{\theta}_k}{\sum_{k=0}^{t-1} \frac{\beta_k}{\gamma_k} - \bar{\theta}_k} = \frac{\alpha_t}{\beta_t}$$

and

$$\frac{1}{\sum_{k=0}^{t-1} \frac{\beta_k}{\gamma_k} - \bar{\theta}_k} = \frac{\gamma_t}{\beta_t}.$$

Without loss of generality, we set

$$\frac{\gamma_t}{\beta_t} = \frac{1}{rS\Omega_t}$$

so that

$$\sum_{k=0}^{t} (\bar{\theta}_k) = \sum_{k=0}^{t} \frac{\beta_k}{\gamma_k} = rS \sum_{k=0}^{t} \Omega_k.$$

and

$$\frac{\alpha_t}{\beta_t} = \frac{\eta\Lambda_t}{rS\Omega_t}$$

so that:

$$\eta \sum_{k=0}^{t} \bar{\theta}_k = \sum_{k=0}^{t} \frac{\beta_k}{\gamma_k}\frac{\alpha_k}{\beta_k} = \sum_{k=0}^{t} \Lambda_k.$$

The system of equations in (15) follows. This system of equations is a fixed point: to solve it, we solve the problem recursively (as in Appendix A.3, except accounting for the rumor) over 4 periods. We then start with guess values for $\{\Omega_j\}_{j=0}^3$ and $\{\Lambda_j\}_{j=0}^3$ and get, through the fixed point in (15), new values for these coefficients. Iterating and invoking the Contraction-Mapping Theorem, we obtain the equilibrium coefficients. ■