Knowledge Cycles and Corporate Investment[*]

M. Cecilia Bustamante[†]   Julien Cujean[‡]   Laurent Frésard[§]

July 10, 2019

Abstract

We propose a theory of how the process of knowledge creation within firms affects their investment decisions. Firms accumulate knowledge through successive rounds of experimentation in the form of capital expenditures, and reset knowledge when they explore new technologies. This process generates endogenous knowledge cycles, which govern firms' investment. Because risky experimentation makes firms information averse, investment increases in knowledge but Q decreases in knowledge. The relationship between investment and Q thus varies over the knowledge cycle and is strongest early in the cycle. We find empirical support for the knowledge channel using a text-based measure of knowledge cycles from public firms. The knowledge channel could explain why investment has been weak in recent years despite high valuation.

Keywords: Experimentation, Exploration, Investment, Knowledge, Information Aversion, Intangibles.

1   Introduction

Over the last decades, intangible resources have become paramount to firms' growth and success. Knowledge capital, such as employees' expertise, proprietary technologies, internal data, organizational design, or customer relationships now represent a central component of firms' assets, almost surpassing the traditional role played by physical capital (e.g., Corrado and Hulten (2010)). Recent research indicates that accounting for such intangibles is key to properly measure and understand firms' investment decisions, especially the abnormally low investment rates observed since the early 2000s (e.g., Gutierrez and Philippon (2017), Peters and Taylor (2017), or Crouzet and Eberly (2018)). Yet, existing research has been surprisingly silent on the economic channels through which knowledge interacts with physical capital, and the resulting implications for firms' investment policy.

In this paper, we study the interactions between firms' creation of knowledge and their investment decisions. Our starting point is to view firms as storehouses of information (e.g., Prescott and Visscher (1980)) and "knowledge" as the accumulated information about the quality of their current technology.[1] We argue that knowledge is a very peculiar type of production input, as it is hard to buy externally, difficult to accumulate and retain (i.e., expert employees), can depreciate fast, and yields highly uncertain output. Thus, unlike physical capital, firms cannot adjust their stock of knowledge directly. Instead, following Arrow (1962), we consider that knowledge arises as a by-product of economic activity. Specifically, we posit that the evolution of knowledge results from successive rounds of risky experiments with the current technology that produce novel information, and the exploration of new technologies that reset existing knowledge.

Embedding this process of knowledge creation in a neoclassical model of investment (e.g., Hayashi (1982)) implies that firms' optimal investment decisions are dictated by endogenous knowledge cycles. Linking firms' investment and knowledge through a trade-off between experimentation and exploration offers several insights that can help interpret recent empirical facts.[2] For instance, weak investment despite high corporate valuation (i.e., high Q), as

---

[1]Our definition of "technology" is generic and could emcompass various forms, such a production process (e.g., artificial intelligence), an organizational design (e.g., decentralized authority, see Prescott and Visscher (1980)), or a management structure (e.g., lean management, see Bloom, Sadun, and van Reenen (2016)).

[2]While the exploration-experimentation trade-off has been used study the innovation process (e.g., Weitzman (1979), March (1991), or Moscarini and Smith (2001), its use in finance has been limited. Manso (2011), Balsmeier, Fleming, and Manso (2011), and Bergemann and Hege (2005) are notable exceptions.

observed in recent years (e.g., Gutierrez and Philippon (2017)), could arise if firms spend more time at earlier stages of their knowledge cycle, a trend recently documented by Hoberg and Maksimovic (2019).

The model focuses on a representative firm that experiments with its technology. The firm is imperfectly informed as it does not observe the quality of this technology. Yet, it accumulates information about it by passively observing productivity realizations. We define "knowledge" as the firm's confidence (in a statistical sense) about the quality of its technology. More knowledge is valuable since it improves the firm's decision. Besides passive learning, the firm can obtain additional knowledge by actively experimenting with its current technology (Thomke (1998)). Specifically, the firm goes through multiple rounds of trial and error until the experiments become conclusive. Similar to Farboodi, Mihet, Philippon, and Veldkamp (2019), experimentation takes the form of capital expenditures (e.g., expanding a research center or equipment with similar technological features). Experiments costly and risky, as their outcome is uncertain. Thus, capital investment affects firm value by altering output (the traditional neoclassical channel), but also by accelerating knowledge accumulation through risky experimentation. We label this effect the "knowledge channel".[3]

At any time the firm can decide to explore new technologies and abandon its current technology. When it does, the quality of the new selected technology is unknown (e.g., Jovanovic and Nyarko (1996)). Switching technologies renders part of the existing stock of physical capital obsolete (e.g., Bresnahan, Greenstein, Brownstone, and Flamm (1996)). Furthermore, when the firm decides to abandon its current technology, it resets its knowledge completely. The firm thus explores when it becomes sufficiently confident that the quality of its existing technology is poor. We further assume that new firms enter the industry as technological quality improves, which reduces profits (Jovanovic, 1982; Ericson and Pakes, 1995). Thus, the firm also explores when it becomes sufficiently confident that the quality of its current technology is high, since future profits will likely be competed away. The trade-off between exploration and experimentation creates endogenous knowledge cycles, within which the firm's stock of knowledge evolves through passive observation and active experimentation, until the firm decides to explore and start a new cycle.

A key feature of the model is that experimentation makes the firm "information averse",

---

[3]In the model, the economic function of the knowledge channel operates as an information acquisition problem. See, for instance, van Nieuwerburgh and Veldkamp (2010) or Hellwig, Kohls, and Veldkamp (2012).

although the firm is risk-neutral when it maximizes its value by choosing investment (whether or not to experiment) and its technology (whether or not to explore). Because the outcome of an experiment is uncertain, it affects firm value only after the firm has made its investment decision. It follows that, depending on the shape of the relation between firm value and knowledge, experimentation is either a gamble or a risk the firm would prefer to avoid. Intuitively, experimentation occurs following exploration when entry by other firms is unlikely and knowledge is scarce. At this early stage of the knowledge cycle, the marginal value of capital and knowledge is highest, the relation between firm value and knowledge is concave, and thus the firm is endogenously information averse (e.g., Andries and Haddad (2019)).

A consequence of information aversion, and our first main result, is that investment is *increasing* in knowledge. To put this result in perspective, consider shutting down the knowledge channel. The resulting model constitutes a "neoclassical benchmark" that delivers the insight from modern $q-$theory that investment is increasing in firm value. Because q is high when knowledge is low, the relation between investment and knowledge is *decreasing* in this neoclassical benchmark. The knowledge channel, however, turns this neoclassical insight on its head. Through the information aversion it creates, the knowledge channel induces the firm to invest less despite high q when knowledge is limited; this generates an increasing relation between investment and knowledge. An empiricist who ignores the presence of the knowledge channel may conclude erroneously that the firm invests too little (compared to what q can justify) following exploration and invests too much as knowledge accumulates.

The knowledge channel renders marginal q difficult to measure and represents a possible origin for its well-known empirical mis-measurement (e.g., Erickson and Whited, 2000). Note first that marginal q is a poor statistic for investment in the model, since it is either unrelated to investment (due to fixed adjustment costs) or negatively related to investment (due to the knowledge channel). Similar to Caballero and Leahy (1996), average Q, which empiricists typically use, does better than marginal q as an explanatory variable for investment. Computing average Q in the model involves observing all possible experimentation outcomes. Because an empiricist does not, she cannot measure Q properly. We demonstrate, however, that realized average Q exhibits little covariation with the knowledge channel, and thus performs much better than marginal q as a statistic for investment.

The second key insight of the model is that the strength of the investment$-Q$ relation depends markedly on the stage of the knowledge cycle. This result is important because

3

absent the knowledge channel (our neoclassical benchmark) the sensitivity of investment to Q is constant throughout the knowledge cycle. In the presence of experimentation the statistical power of Q in explaining investment varies over the cycle. For instance, when entry costs are low the statistical power of Q is strongest at intermediate stages of the cycle, and shifts to early stages of the cycle as entry costs rise. Intuitively, increased competitive threats cause the firm to experiment more often, and to invest more aggressively as knowledge accumulates. Improving the amount of information gathered from experiments has similar effects.

We test the novel implications of the model using the text-based measures of product life cycles recently developed by Hoberg and Maksimovic (2019), who decompose product cycles into four distinct stages: product innovation, process innovation, maturity and decline. We argue that these four stages are reasonable proxies for firms' position in their knowledge cycle, and report evidence supporting the empirical relevance of the knowledge channel. In particular, in line with Hoberg and Maksimovic (2019), physical investment increases with knowledge in the data. Investment is weak following exploration (in the first stage of the cycle) and rises as knowledge accumulates (in later stages). In contrast, Q is decreasing in knowledge, peaking following exploration and declining later in the cycle.

The measured sensitivity of investment to Q varies significantly over the knowledge cycle. It is negative after exploration, and turns positive and strong in the second and third stage of the knowledge cycle. These patterns are consistent with the model's predictions, and contrast with those of our neoclassical benchmark. Similar to Hoberg and Maksimovic (2019), we further document an aggregate "shortening" of the knowledge cycle, with firms exploring more intensively on average (i.e., being more frequently in the first stage). The aggregate rise in firms' exploration intensity could be due, for instance, to an increased difficulty to find high quality inventions as recently noted by Bloom, Jones, van Reenen, and Webb (2017) or to more costly and risky experimentation, both forces inducing firms to explore more frequently. The model predicts that this shortening should result in weaker investment despite high Q, facts recently documented and studied by Gutierrez and Philippon (2017), Alexander and Eberly (2018), and Crouzet and Eberly (2018), and also present in our sample.

This paper adds to a growing literature studying the role of knowledge-related assets in firms' investment decisions. Eisfeldt and Papanikolau (2013) and Peters and Taylor (2017) introduce intangible capital in addition to physical capital in a neoclassical investment model,

assuming that the two stocks of capital do not interact. Closer to this paper, Andrei, Mann, and Moyen (2018) develop an investment model in which the firm learns passively and innovation occurs randomly. We consider active experimentation through investment as a reinforcing learning mechanism, endogenous innovation through the possibility of exploring new technologies, and focuses on the relation between knowledge and investment.

This paper belongs to a large literature on experimentation, and is closest to Moscarini and Smith (2001).[4] Like their decision maker, our firm decides how much to experiment and when to take a payoff-relevant action (permanent stop in their case, and exploration in ours). Like their experimentation level, investment is increasing in knowledge and is highest when the action is about to be taken. A first difference is in what dictates this pattern. Because the cost of delaying action is increasing in value, in their case experimentation must grow in value to raise the surplus from information production. However, this surplus is uncertain in this paper and creates information aversion, which dictates higher investment despite lower value. Another difference is in the formulation and scope. They focus on a problem of sequential analysis (Wald, 1947), whereas we study investment in a neoclassical context.

2 Model setup

We present a model of knowledge as a by-product of economic activity. A firm has imperfect information about the technology it operates, and can choose between experimenting with the technology to learn about its quality or abandoning it to explore a new unknown technology. The tradeoff between exploration (Jovanovic and Rob, 1990; Jovanovic and Nyarko, 1996) and experimentation (Keller and Rady, 1999; Moscarini and Smith, 2001) takes place in the context of a neoclassical model of investment (e.g., Hayashi (1982)).

---

[4]See, for instance, Grossman, Kihlstrom, and Mirman (1977), Weitzman (1979), Jovanovic and Rob (1990), Rob (1991), Aghion, Bolton, Harris, and Jullien (1991), Jovanovic and Nyarko (1996), Keller and Rady (1999), Bergemann and Hege (2005), Callander (2011), Balsmeier et al. (2011), and Manso (2011).

2.1 Knowledge, exploration and experimentation

Consider a firm that combines physical capital, K_t, with some technology to produce a unique non-storable output, Y_t, at time t according to the production function:

$$Y_t = A_{n,t} K_t^{\alpha}, \tag{1}$$

where $A_{n,t}$ denotes the productivity of technology n and $\alpha \in (0,1)$ denotes returns to scale on physical capital. We use the concept of technology in a broad sense as a "technology" may refer for instance to management practices (e.g., Bloom et al. (2016)) or organization designs (e.g., Prescott and Visscher (1980)). At every date t, the firm can choose to continue operating the current technology n about which it has accumulated knowledge, or to abandon it and explore instead by selecting another unknown technology, say $n+1$. Each technology n is characterized by the growth rate of its productivity, M_n, which reflects its quality. The quality of each technology is unobservable and, upon selecting it, is drawn randomly according to:

$$M_{n+1} \sim \mathcal{N}(0, \tau_M^{-1}), \quad \text{with } M_n \perp M_{n+1}, \quad \forall n \in \mathbb{N}.$$

For simplicity, we assume that the quality of each technology is independent of the others, and thus the technology index n can be discarded.[5] In other words, the firm "learns and forgets" (e.g., Benkard (2000)). For later use, we denote by ν any date at which the firm decides to abandon the current technology and explore an unknown technology.

Although the firm does not observe the quality of its current technology, it observes realizations of its productivity. Each incremental realization, $\mathrm{d}A/A$, reveals noisy information about M according to:

$$\mathrm{d}A_t/A_t = \tau_A^{1/2} \frac{M}{\Omega_t^{1/2}} \mathrm{d}t + \mathrm{d}B_t, \tag{2}$$

where B is a Brownian noise and Ω_t denotes the conditional variance of the firm's estimate of quality at time t (defined below). To simplify the interpretation of the model, we scale

---

[5] Alternatively, there is no "standing on the shoulders of giants". This assumption along with the specification of the signals in Eqs. (2)–(3) imply that the parameter τ_M, which determines prior precision regarding the quality of the technology, will play no role in the analysis.

quality, M, by the conditional variance, Ω, so that the conditional variance of productivity realizations is normalized to one. In other words, from the firm's perspective, uncertainty about productivity is fixed. It follows that the precision parameter τ_A measures the informativeness of the productivity signal (about M) in time-invariant units of uncertainty.[6]

In addition to observing realizations of productivity, the firm can also learn actively about M through experimentation, or "learning by doing" (Arrow, 1962; Grossman et al., 1977)). The firm experiments by investing additional capital in the current technology. Formally, let $I_t \geq 0$ represent investment in physical capital at time t and $i_t = I_t/K_{t-}$ be the corresponding investment rate. By investing (i.e., $i_t > 0$) the firm receives an informative signal flow about the quality of the current technology:

$$\mathrm{d}S_t = \mathbf{1}_{i_t > 0} \left(\frac{\tau_S^{1/2} i_t^{1/2}}{\Omega_{t-}^{1/2}} M + \epsilon_t \right), \quad \epsilon_t \sim \mathcal{N}(0,1) \text{ and i.i.d.} \tag{3}$$

Investment improves the precision of this signal, so that the more the firm invests relative to its current stock of capital, the more it learns about M. However, when the firm experiments by investing, the outcome is uncertain, an aspect we model using *experimentation noise*, ϵ. Similar to the passive productivity signal, informativeness of $\mathrm{d}S$ is measured in time-invariant units of uncertainty. To further simplify intuition, we rule out learning through capital liquidation by making investment strictly irreversible.

The firm's information about the quality of the technology it operates entirely results from observing the passive and active signals:

$$\mathscr{F}_t = \sigma \left((A_s, S_s) : s \leq t \right).$$

Based on this information, the firm computes an estimate, $\widehat{M_t} = \mathbb{E}[M | \mathscr{F}_t]$, of its quality. The conditional error variance of this estimate is $\Omega_t = \mathbb{V}[M | \mathscr{F}_t]$ (introduced above). Standard

---

[6]See Kyle, Obizhaeva, and Wang (2018) for a similar formulation of signals, albeit in a different context.

arguments imply that the firm gradually updates these statistics according to:

$$d\widehat{M}_t = \Omega_t^{1/2}\tau_A^{1/2}d\widehat{B}_t + \Omega_{t-}^{1/2}\left(\frac{\tau_S i_t}{1+\tau_S i_t}\right)^{1/2}\widehat{\epsilon}_t\mathbf{1}_{i_t>0} - \widehat{M}_{t-}\mathbf{1}_{t=\nu}$$

$$d\Omega_t = -\Omega_t\tau_A dt - \Omega_{t-}\frac{\tau_S i_t}{1+\tau_S i_t}\mathbf{1}_{i_t>0} + (\tau_M^{-1} - \Omega_{t-})\mathbf{1}_{t=\nu},$$

where $\widehat{B}$ is a Brownian motion and $\widehat{\epsilon}$ is Gaussian with mean zero and unit variance under the firm's probability measure.[7] In short, the firm updates its estimates about the quality of its current technology continuously by observing realized productivity, punctually by experimenting with it, and resets its estimate to 0 and its prior precision to τ_M (i.e., its prior beliefs) whenever it decides to explore another technology.

Let the firm's stock of knowledge (i.e., intangible capital), Z_t, represent the information it has at date t about the technology it operates. In the context of the model, the firm's estimate, $\widehat{M}$, and its standard error, $\Omega^{1/2}$, do not matter separately; they only matter as a t-statistic ratio, $Z \equiv \widehat{M}/\Omega^{1/2}$, which summarizes the firm's stock of knowledge:

$$dZ_t = \underbrace{\tau_A/2Z_t dt + \tau_A^{1/2}d\widehat{B}_t}_{\text{passive learning}} + \underbrace{Z_{t-}\left((1+\tau_S i_t)^{1/2}-1\right) + (\tau_S i_t)^{1/2}\widehat{\epsilon}_t}_{\text{learning by experimenting}} \underbrace{-Z_{t-}\mathbf{1}_{t=\nu}}_{\text{knowledge reset}} . \quad (4)$$

Positive (negative) values of Z indicate that the firm is confident in a statistical sense that the quality of its current technology is high (low). On average knowledge moves away from prior beliefs (i.e., $Z_0 = 0$), accumulating in continuation of the current estimate, Z_t, at the speed with which the passive and active signals reveal information (i.e., $\tau_A/2$ and $\approx \tau_S i_t/2$, respectively). Exploring a new technology resets the firm's knowledge to prior beliefs (i.e., $Z_\nu = 0$). The stock of knowledge is the key variable based on which the firm decides to experiment or explore.

2.2 The decision of the firm

We now turn to the firm's value function and its decision problem. Suppose the firm operates in an industry with a continuum of firms that produce the same good using the same tech-

---

[7]See Theorem 12.7 in Lipster and Shiryaev (2001) for the continuous part, and Theorem 7.1 Jazwinski (1970) for the discontinuous part.

nology. Although we do not intend to construct an industry equilibrium, it is conceptually helpful to give this industry some structure, the description of which follows Caballero and Pindyck (1996). The mass of firms in the industry at instant t is N_t. Indexing firms by i, total industry output is:

$$Q_t \equiv \int_0^{N_t} Y_{i,t} \mathrm{d}i.$$

We decompose individual firm output, Y_i, into an aggregate component (average output in the industry), Y, which we model as in Eq. (1), and a firm-specific component, y_i, as $Y_{i,t} = Y_t y_{i,t}$ at date t; we leave the component y_i unmodeled but assume that it satisfies:

$$\int_0^{N_t} y_{i,t} \mathrm{d}i = N_t.$$

Hence, total output in the industry is $Q_t \equiv N_t Y_t$.

We take the industry demand curve to be isoelastic:

$$P_t = (Y_t N_t)^{-\eta},$$

where $\eta \in (0,1)$ is interpreted as the price elasticity of demand. We then use this demand curve to construct the revenues of the average firm in the industry:

$$\Pi(A_t, K_t, N_t) = P_t Y_t = A_t^{1-\eta} K_t^{\alpha(1-\eta)} N_t^{-\eta}.$$

Revenues of the average firm are increasing in the productivity of the current technology, A, and decreasing in the number of firms in the industry, N. From now onwards, we refer to "the firm" as the average firm in the industry, and focus on it exclusively; accordingly, considering the variables of the previous section, we interpret A as aggregate productivity of the technology, K as average physical capital, and Z as average knowledge in the industry about the current technology that all firms operate.

The firm maximizes profits (revenues net of costs) by optimally choosing investment and its technology. Every time the firm decides to invest—times we denote by θ—it increases its stock of physical capital and knowledge. However, it also incurs adjustment costs, $\gamma_t(i)$, for purchasing and installing new capital, which we assume proportional to operating revenues

(e.g., Cooper (2006); Hackbarth and Johnson (2015)):

$$\gamma_t \equiv \left(\kappa + \gamma/2i^2\right) \cdot \Pi\left(A_t, K_t, N_t\right),$$

where κ represents a fixed cost and the variable cost is convex (e.g., Abel and Eberly (1994)).[8] Similarly, abandoning current technology to explore is costly. At the time the firm explores, we assume that a fraction ω of the firm's physical capital K becomes obsolete, i.e, exploration involves a lump-sum cost of obsolescence, ωK. We further assume that physical capital depreciates at fixed proportional rate, δ. It follows that the net change in the firm's stock of physical capital is:

$$\mathrm{d}K_t/K_t = i_t \mathbf{1}_{t=\theta} - \delta \mathrm{d}t - \omega \mathbf{1}_{t=\nu}.$$

The stock of knowledge, Z, thus entirely determines the evolution of physical capital, K.

Whereas the firm has clear incentives to explore when the quality of the current technology is likely poor, without further assumptions, the firm has none when it is likely high. We model in reduced form a common result in the literature (Jovanovic, 1982; Ericson and Pakes, 1995) that new firms enter the industry as technological efficiency—in our case, knowledge about the quality of the technology—improves. The resulting increase in the mass of firms in the industry, N, erodes operating revenues and thus creates an incentive for the firm to explore. Formally, we assume that the mass of firms evolves according to:

$$\mathrm{d}N_t/N_t = \phi Z_t^2 \mathrm{d}t, \tag{5}$$

where ϕZ_t^2 corresponds to the entry rate at time t.[9] The main implication is that competitive pressures keep knowledge stationary in the model.

Given the costs and benefits associated with exploration and experimentation, and as-

---

[8]Because the firm value (defined later) exhibits concave and convex regions, variable costs are needed to keep the investment rate finite in the convex regions.

[9]This specification of the entry rate is not necessarily consistent with aggregation of firms' individual decisions, which we do not model. Endogenizing the entry rate would require assumptions on firm-specific shocks, y_i, and the problem of entrants. Our focus, however, is on a representative firm, as opposed to explicitly modeling heterogeneity across firms. Furthermore, this specification implies that the same mechanism operates when the quality of the technology is poor. The intuition of the model is, however, unaffected since the firm has strong incentives to explore when the quality is likely poor, even absent new entry.

suming that the firm is risk neutral and discounts its profits at constant rate, r, its value, $V(\cdot)$, is:

$$\underbrace{\max_{\{\nu_n\},i_{t+s}} \mathbb{E}\left[\int_0^\infty e^{-rs}\Pi\left(A_{t+s},K_{t+s},N_{t+s}\right)\mathrm{d}s - \sum_{n\geq 0}\gamma_{t+\theta_n}e^{-r\theta_n}\,\middle|\,\mathscr{F}_t\right]}_{\equiv V\left(N_t,A_t,K_t,\widehat{M}_t,\Omega_t\right)}. \qquad (6)$$

Our assumptions further imply that the firm value has the convenient representation:

$$V\left(N_t,A_t,K_t,\widehat{M}_t,\Omega_t\right) \equiv \Pi\left(A_t,K_t,N_t\right)\cdot v\left(\widehat{M}_t/\Omega_t^{1/2}\right) \equiv \Pi\left(A_t,K_t,N_t\right)\cdot v\left(Z_t\right), \qquad (7)$$

where $v(\cdot)$ denotes the intensive value of the firm. This representation confirms that the firm's knowledge, Z, is the only variable that determines its choice between exploration and experimentation.

3   Trading off exploration against experimentation

The main feature of the model is that knowledge arises as a by-product of economic activity. Throughout the analysis we define this "knowledge channel" as follows.

Definition 1. (*Knowledge channel*) *By "knowledge channel" we refer to the channel through which investment affects firm value (on average) by marginally improving knowledge, ignoring the marginal contribution of physical capital. Formally, the knowledge channel satisfies:*

$$c(Z) \equiv \int_{\mathbb{R}} \frac{V(\cdot,K+I,x)}{\Pi(\cdot)}\frac{\mathrm{d}}{\mathrm{d}i}\varphi(x;i(Z),Z)\,dx, \qquad (8)$$

where $\varphi(\cdot;i(Z),Z)$ denotes the normal density with mean $\sqrt{1+\tau_S i(Z)}Z$ and variance $\tau_S i(Z)$ (i.e., the distribution of incremental knowledge gathered from investing, as per Eq. (4)).

This definition can be equivalently expressed in terms of intensive firm value as:

$$c(Z) = (1+i(Z))^{\alpha(1-\eta)}\int_{\mathbb{R}} v(x)\frac{d}{di}\varphi(x;i(Z),Z)\mathrm{d}x.$$

11

It represents the sensitivity to investment of what the firm expects to be worth upon investing, keeping the marginal effect of physical capital fixed. To study how the knowledge channel affects neoclassical predictions, it is convenient to first present a benchmark model in which it is absent. We then solve the full model.

3.1   A benchmark model of exploration without experimentation

We start by shutting down the knowledge channel through which the firm "learns by investing". This is achieved by assuming that experiments through physical capital expansion are uninformative, $\tau_S \equiv 0$. Because investment does not affect knowledge in this case, knowledge becomes irrelevant in the timing of the investment decision. We thus assume that investment occurs continuously, $\kappa \equiv 0$. The problem of the firm simplifies to:

$$\max_{\{\nu_n\},i_{t+s}} \mathbb{E}\left[\int_0^\infty e^{-rs}\left(1 - \gamma/2i_{t+s}^2\right)\Pi\left(A_{t+s}, K_{t+s}, N_{t+s}\right)\mathrm{d}s\right],$$
$$\text{s.t. } \mathrm{d}K_t = K_t((i_t - \delta)\mathrm{d}t - \omega\mathbf{1}_{t=\nu}),$$
$$\mathrm{d}Z_t = \tau_A/2Z_t\mathrm{d}t + \tau_A^{1/2}\mathrm{d}\widehat{B}_t - Z_{t-}\mathbf{1}_{t=\nu},$$

with A and N satisfying the dynamics in Eqs. (2) and (5).

The firm's decision to explore is a standard optimal stopping problem: times, $\{\nu_k\}_{k=0}^\infty$, at which the firm explores correspond to hitting times of knowledge at fixed trigger levels. Given the structure of the model, there exist two exploration triggers, say $\underline{a} < \overline{a}$. Intuitively, the firm has incentives to explore when it is confident that the quality of the technology is poor, $Z \geq \overline{a}$, or when it is likely high thus attracting new entrants, $Z \geq \overline{a}$. Denoting by $\mathscr{A} = (\underline{a}, \overline{a})$ the region in which the firm does not explore, it follows that:

$$\nu_n = \inf\{t \geq \nu_{n-1} : Z_t \notin \mathscr{A}\}, \quad \forall n \in \mathbb{N}. \tag{9}$$

When the firm does not explore, $Z \notin \mathscr{A}$, its intensive value, v, satisfies the HJB equation:

$$v(Z)\left((1-\eta)(\alpha\delta + \eta/2) - (1-\eta)\sqrt{\tau_A}Z + r + \eta\phi Z^2\right) \tag{10}$$
$$= 1 + ((1-\eta)\sqrt{\tau_A} + \tau_A/2Z)\,v'(Z) + \tau_A/2v''(Z) + \max_{i\geq 0}\left\{\alpha(1-\eta)v(Z)i - \gamma/2i^2\right\}.$$

Eq. (10) resembles its neoclassical counterpart to the extent that it delivers the insight from modern $q-$theory, namely that investment is an increasing function of firm value:

$$i_t \equiv i(Z) = \frac{\alpha(1-\eta)}{\gamma} v(Z). \tag{11}$$

However, a meaningful way in which Eq. (10) differs from the neoclassical equation is that the force driving firm value and thus investment is knowledge, as opposed to physical capital. The point is that shutting down the knowledge channel reproduces the neoclassical relation between firm value and invesment; yet it does not bring us back to a neoclassical framework since investment depends on knowledge, which is usually abstracted away from in a neoclassical framework.

In this model, the evolution of knowledge is distinct from the way physical capital accumulates and depreciates in most neoclassical models. Knowledge accumulates in cycles (defined below) and depreciates at once when the firm explores. In the absence of experimentation through invesment (the benchmark model of this section), knowledge accumulation within cycles is not materially different from physical capital accumulation in traditional models; what makes its dynamics distinct is the necessity for the firm to explore. When a new technology is adopted, the firm learns passively about its productive potential. After some time, however, the learning potential on this technology will decline, and so will the firm's profits. Exploration is necessary for the firm to regenerate future profits.

Formally, exploration acts as a centripetal force on knowledge—it resets knowledge to priors. Upon exiting the region $\mathscr{A}$ it is optimal for the firm to explore. As a result of exploration, the firm resets its knowledge to priors and incurs a lump-sum cost of obsolescence on its existing capital stock, ωK. Hence, the intensive firm value upon exploration is:

$$v\left(Z\right) = (1-\omega)^{\alpha(1-\eta)} v\left(0\right), \;\; Z \notin \mathscr{A}. \tag{12}$$

The firm goes through multiple rounds of exploration, which we refer to as "knowledge cycle" in a way that we define below.

Definition 2. *(Knowledge Cycle)* *A knowledge cycle starts when the firm resets its stock of knowledge to 0 by exploring a new technology (i.e., $Z_0 = 0$), and ends after a period of (random) length ν (as defined in Eq. (9)) when the firm abandons the technology to explore*

13

yet another new technology (i.e., $Z \notin \mathscr{A}$).

We conclude that the intensive value of the firm, v, satisfies Eq. (10) when the firm chooses not to explore and Eq. (12) otherwise. We finally determine the optimal exploration thresholds through "smooth pasting". Specifically, the firm chooses optimally when to explore (i.e., the two optimal exploration triggers, $\underline{a}$ and $\bar{a}$) according to:

$$\lim_{Z \searrow \underline{a}} v'(Z) = \lim_{Z \nearrow \bar{a}} v'(Z) = 0. \tag{13}$$

We then solve for the firm value numerically, using the baseline parameter values summarized in Table 1 (on which we elaborate in the next subsection).[10] In Figure 1 we plot the firm's value as a function of the firm's stock of knowledge. Since investment is proportional to firm value in the benchmark model, we may talk indifferently of firm value and investment when describing Figure 1.
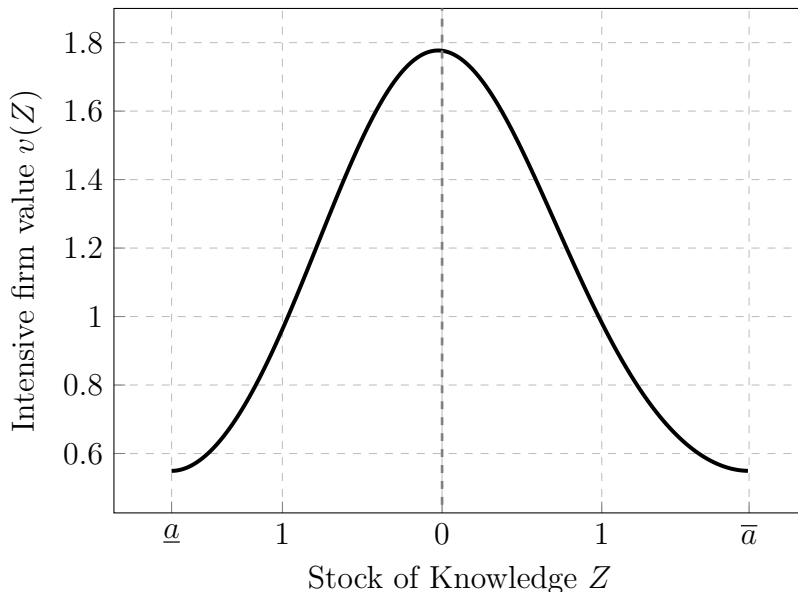

Figure 1: Firm value as a function of knowledge in the benchmark model. The figure plots intensive firm value as a function of knowledge, Z. Smooth-pasting for exploration requires that firm value lands flat at the edges of the domain defined by the exploration thresholds $\underline{a}$ and $\bar{a}$. Baseline parameter values are defined in Table 1.

---

[10]The only difference relative to Table 1 (besides $\tau_S \equiv \kappa \equiv 0$) is that we increase the value of γ to 0.3 to make up for the absence of fixed costs.

The main takeaway from inspecting Figure 1 is that firm value and thus investment are hump-shaped in knowledge. They are highest following recent exploration (i.e., $Z \approx 0$), and weakest when exploration is imminent (i.e., $Z \approx \underline{a}$ or $Z \approx \overline{a}$). Intuitively, periods immediately following exploration are associated with high marginal product of capital due to lower competitive pressures, and with high marginal product of knowledge due to the high learning potential on the new technology.[11] We now build on these insights to incorporate experimentation in the model.

3.2   Model solution in the presence of the knowledge channel

In this model, experimentation works as follows. Experimentation is optimal within an intermediate range, $\mathscr{B} = [\underline{b}, \overline{b}]$, of knowledge. Whenever the firm's knowledge enters this range, the firm starts experimenting. Since the outcome of the experiment is uncertain, the firm may end up re-entering the experimentation range after the first trial. The firm will typically go through multiple rounds of trial and error until the experiment eventually becomes conclusive (in the sense that $Z \notin \mathscr{B}$). Furthermore, because the firm does not know ex-ante the experiment outcome, it must decide on a "knowledge-contingent investment plan", $i(Z)$, for every possible outcome that falls within the experimentation range. Finally, in this model, experimentation and exploration alternate endogenously, as opposed to following a pre-determined schedule (e.g., Berk, Green, and Naik (2004); Pastor and Veronesi (2009)).

Formally, we conjecture that experimentation is optimal at early to intermediate stages of the cycle, $\mathscr{B} \subset \mathscr{A}$. Using this notation, under fixed adjustment costs investment will occur in lumps, I, at times:

$$\theta_n = \inf\{t \geq \theta_{n-1} : Z_t \in \mathscr{B}\}, \quad \forall n \in \mathbb{N}.$$

Building on the intuition gathered from the benchmark model, we further conjecture that experimentation will take place in a neighborhood of the knowledge reset, $\{0\} \in \mathscr{B}$, where the marginal product revenue on capital and knowledge is highest.

We start by solving for the value of the firm when it neither explores nor experiments.

---

[11]The symmetry between good and bad technological outcomes is an artifact of the calibration that assumes high competitive pressures. Reducing competitive pressures makes the exploration cycle longer, increases investment throughout the cycle, and creates an asymmetry whereby likely poor technologies have a shorter cycle and likely high-quality technologies have a longer cycle.

We denote by $\mathscr{I}$ an arbitrary inaction region and by $g(\cdot)$ the intensive value of the firm when inactive; we highlight its general solution in the proposition below (see Appendix A).

Proposition 1. *The solution to the intensive firm value in an inaction region, $Z \in \mathscr{I}$, is:*

$$g\left(Z; C_1, C_2\right) := v_P\left(Z\right) + e^{g_1 Z + g_2 Z^2}\left(C_1 H_n\left(h_0 + h_1 Z\right) + C_2 M\left(-\frac{1}{2}n, \frac{1}{2}, m_0 + m_1 Z + m_2 Z^2\right)\right),$$
(14)

where C_1 and C_2 are two integration constants, $v_P(\cdot)$, is a particular solution:

$$v_P(Z) = \int_0^\infty \exp\left(a_0(s) + a_1(s)Z + a_2(s)Z^2\right) ds,$$
(15)

the function, $H_n(\cdot)$, represents the Hermite polynomial of order:

$$n = \frac{1}{2}\left(\frac{2(\eta - 1)^2 \tau_A - \xi^2(-2(\eta - 1)(2\alpha\delta + 1) + 4r + \tau_A)}{\xi^3 \sqrt{\tau_A}} - 1\right),$$
(16)

and $M(\cdot)$ denotes the the Kummer confluent hyper-geometric function. The definitions of the functions $a_0(\cdot)$, $a_1(\cdot)$ and $a_2(\cdot)$, and of other parameters are presented in the appendix.

There are two inaction regions in the model, $(\underline{a}, \underline{b})$ and $(\overline{b}, \overline{a})$. The general solution of Proposition 1 only differs across the two regions by integration constants. We use the notation $\underline{C}_1$ and $\underline{C}_2$ for the two constants over the lower region, and $\overline{C}_1$ and $\overline{C}_2$ over the upper region. We now turn to the solution of the intensive firm value over the regions in which the firm either explores or experiments.

In this model, experimentation acts as a centrifugal force on knowledge—it drives knowledge away from priors. Upon entering the experimentation region, $\mathscr{B}$, the firm will keep experimenting until it becomes sufficiently confident (in the sense that $Z \notin \mathscr{B}$) that the current technology is either good or bad. That is, as the firm starts experimenting, it needs to anticipate that each experiment may bring knowledge back in the experimentation region, thus leading to another round of experimentation. Importantly, for each realization of knowledge, $Z \in \mathscr{B}$, that falls within the experimentation region, the firm will optimally choose a different investment policy. In other words, the firm will re-adjust its investment rate depending on the outcome of the experiment. Thus not only does the firm need to anticipate future

16

rounds of experimentation, it also needs to determine an optimal knowledge-contingent plan, $i(Z)$, for investment at each experimentation round.

Formally, let $Z^{(0)}$ be the stock of knowledge upon first entry in the experimentation region, and $i(Z^{(0)})$ be the associated investment choice. Based on the knowledge dynamics in Eq. (4), this first experiment will update knowledge to a new (random) level:

$$Z^{(1)} \equiv Z^{(0)}(1 + \tau_S i(Z^{(0)}))^{1/2} + \sqrt{\tau_S i(Z^{(0)})}\,\widehat{\epsilon}^{(1)}. \tag{17}$$

Suppose further that the firm goes through n such experimentation rounds. Since each round of experimentation takes place instantaneously (each round has Lebesgue measure zero), knowledge, $Z^{(n)}$, after the $n-$th round of experimentation satisfies:

$$Z^{(n)} = Z^{(0)} \prod_{m=0}^{n-1}(1 + \tau_S i(Z^m))^{1/2} + \sum_{k=1}^{n-1} \sqrt{\tau_S i(Z^{(k-1)})}\,\widehat{\epsilon}^{(k)} \prod_{m=k}^{n-1}(1 + \tau_S i(Z^{(m)}))^{1/2} + \sqrt{\tau_S i(Z^{(n-1)})}\,\widehat{\epsilon}^{(n)}. \tag{18}$$

Eq. (18) shows that, across rounds of experimentation, knowledge evolves as an AR(1) process, the persistence and volatility of which are endogenously determined by investment. Based on this law of motion, we define an "experimentation round" as the number of trials it takes until experimentation becomes conclusive in a statistical sense.

Definition 3. (*Experimentation Round*) *A round of experimentation starts upon first entry in the experimentation region, $Z^{(0)} \in \mathscr{B}$, and ends after $n-$th trial:*

$$n = \inf\{k \in \mathbb{N} : Z^{(k)} \notin \mathscr{B}\}$$

upon first exit of the experimentation region, where $Z^{(k)}$ is defined in Eq. (9).

The firm must determine a knowledge-contingent policy for investment when experimentation active. In particular, letting $\mathscr{D}$ denote the feasible investment set (to be defined below), the firm solves:

$$V(Z) = \sup_{i \in \mathscr{D}} \underbrace{(1+i)^{\alpha(1-\eta)}}_{\text{returns on physical capital}} \underbrace{\int_{\mathbb{R}} v(x)\varphi(x; i(Z), Z)\mathrm{d}x}_{\text{expected returns on knowledge}} - \kappa - \gamma/2i^2, \quad Z \in \mathscr{B}, \tag{19}$$

where $\varphi(\cdot; i(Z), Z)$ is the distribution of incremental knowledge built from experimentation (as defined in Definition 1). The expectation term represents the knowledge channel that is specific to this model and results from experimentation. Because the outcome of experimentation is uncertain (it involves noise), the firm does not know *ex-ante* what information it will get as a result of experimentation. What the firm knows is that experimentation through active investment improves knowledge on average. Furthermore, the possibility of successive rounds of experimentation makes the investment problem in Eq. (19) a dynamic one; it will be reformulated in the form of a Bellman equation shortly.

As in the benchmark model, and in contrast to experimentation, exploration resets knowledge to priors. In this case, the intensive firm value in the exploration region is:

$$v(Z) = (1 - \omega)^{\alpha(1-\eta)} V(0), \quad Z \notin \mathscr{A}. \tag{20}$$

The difference relative to exploration in the benchmark model (i.e., Eq. (12)) is that exploration leads to immediate experimentation, and is thus associated with firm value, $V(0)$, when experimenting. We conclude that the intensive firm value can be written piecewise as:

$$v(Z) = \begin{cases} g(Z; \underline{C}_1, \underline{C}_2), & Z \in (\underline{a}, \underline{b}) \\ V(Z), & Z \in \mathscr{B} \\ g(Z; \overline{C}_1, \overline{C}_2), & Z \in (\overline{b}, \overline{a}) \\ (1 - \omega)^{\alpha(1-\eta)} V(0), & Z \notin \mathscr{A} \end{cases}. \tag{21}$$

Based on this piecewise representation we reformulate the investment problem in Eq. (19) recursively. It is convenient to denote by $h(Z)$ firm value outside the experimentation region $\mathscr{B}$:

$$h(Z) \equiv g(Z; \underline{C}_1, \underline{C}_2) \mathbf{1}_{Z \in (\underline{a}, \underline{b})} + (1 - \omega)^{\alpha(1-\eta)} V(0) \mathbf{1}_{Z \notin \mathscr{A}} + g(Z; \overline{C}_1, \overline{C}_2) \mathbf{1}_{Z \in (\overline{b}, \overline{a})}.$$

We highlight the recursive formulation of Eq. (19) in the proposition below.

Proposition 2. *The value function, $V(Z)$, defined in Eq. (19), associated with optimal*

18

experimentation satisfies the Bellman equation:

$$V(Z) = \sup_{i \in \mathscr{D}} (1+i)^{\alpha(1-\eta)} \int_{\mathbb{R}} h(y)\varphi(y; i(Z), Z) \, dy - \kappa - \gamma/2i^2 \qquad (22)$$
$$+ (1+i)^{\alpha(1-\eta)} \int_{\mathscr{B}} V(y)\varphi(y; i(Z), Z) \, dy,$$

where the admissible investment set is given by:

$$\mathscr{D} = \left[0, \frac{2}{\tau_S(\sqrt{\tau_A + 8\eta\phi}/\sqrt{\tau_A} - 1)} \right).$$

The Bellman equation (22) formalizes the firm's dynamic investment problem across experimentation rounds. At every experimentation round, the firm trades off adjustment costs against the marginal revenue product of knowledge and capital, which the firm re-evaluates depending on the updated likelihood of each experimentation outcome. Through the first integral the firm anticipates that experimentation may lead to future inaction or immediate exploration; through the second integral the firm anticipates that experimentation may lead to yet another round of experimentation, which precisely makes the problem recursive across successive rounds of experimentation.

The rest of the solution method is standard. We must first impose continuity across the five smoothness regions, which gives us four "value-matching conditions" to determine the four constant of integration, $\underline{C}_1$, $\underline{C}_2$, $\overline{C}_1$, and $\overline{C}_2$. A technical difficulty at this stage is that the Bellman equation (22) does not have an explicit solution. In Appendix C, we explain how to proceed numerically. We then choose each threshold optimally by imposing that the firm value be continuously differentiable across regions. As in the benchmark model, the firm chooses optimally when to explore according to Eq. (13). Similarly, the firm chooses optimally when to experiment (i.e., the two experimentation triggers, $\underline{b}$ and $\overline{b}$) according to:

$$\lim_{Z \nearrow \underline{b}} g'(Z; \underline{C}_1, \underline{C}_2) = \lim_{Z \searrow \underline{b}} V'(Z), \qquad (23)$$
$$\lim_{Z \nearrow \overline{b}} V'(Z) = \lim_{Z \searrow \overline{b}} g'(Z; \overline{C}_1, \overline{C}_2),$$

Eq. (23) implies that the firm experiments until the marginal benefit of one additional unit of knowledge on firm value when inactive and on its expected value when experimenting are

equalized.

From now onwards, we proceed numerically. It is convenient to define baseline parameter values, which we gather in Table 1. We start with the parameters that do not influence our main results: we normalize the informativeness of the productivity signal to $\tau_A \equiv 1$, and set the discount rate to $r \equiv 0$ (the rate of entry parameter, ϕ, already acts as a discount rate). We further set variable adjustment costs to $\gamma = 5\%$ (the concavity of firm value in the experimentation region is the main force keeping investment bounded), and assume that capital only depreciates through obsolescence (i.e., $\delta \equiv 0$; given the homogeneity of the problem, continuous depreciation merely acts as another discount rate).

Importantly, conjecturing that firm value is concave when experimentation is optimal, "expected returns on knowledge" will be endogenously lower than $1/2$ (by Jensen's inequality and Bayes' rule); accordingly, we choose returns on physical capital, $\alpha(1-\eta) = 0.3 < 1/2$, so that they are of comparable magnitude. This aspect will play a role in the relation between knowledge and investment. Another key parameter in determining the behavior of optimal investment is how informative experiments are. Specifically, to ensure that investment is interior to $\mathscr{D}$, experiments need to be sufficiently noisy (when variable adjustment costs are low); intutively, the firm could otherwise resolve uncertainty at once, causing experimentation and exploration to alternate simultaneously. But the presence of fixed adjustment costs, which we set to represent $\kappa = 50\%$ of profits, would imply instant ruin. We thus assume that $\tau_S = 1/5$ so that experiments are "noisy" relative to aggregate productivity shocks.

Similarly, it is important to ensure that exploration is sufficiently infrequent and thus that the fixed experimentation cost is not paid "too often" by setting obsolescence costs sufficiently high. We assume that $\omega = 99\%$ of the stock of capital becomes obsolete upon exploration. This means that firm value decreases by $1 - (1 - \omega)^{\alpha(1-\eta)} \approx 75\%$ upon exploring. Finally, the entry rate into the industry captures competitive pressures and thus how desirable exploration is; it also acts as a discount rate, and determines the asymmetry between good and poor technologies. We set it equal to $\phi = 2$, a high value that triggers approximate symmetry across technologies.

In Figure 2 we illustrate the firm value as a function of knowledge under baseline parameter values. As in the benchmark model, the firm value is highest following recent exploration, thus confirming that the benefits of investing at early stages are highest. In particular, early investment is associated with high marginal revenue of knowledge—the firm has little knowl-

Table 1: Baseline parameter values. This table summarizes parameters that are standard in neoclassical models (part 1), and parameters that are key to the main results (part 2).

Symbol	Definition	Value
1. standard parameters		
τ_A	informativeness of aggregate productivity	1
r	risk-free rate	0
γ	variable adjustment costs	5%
δ	depreciation rate	0%
η	price elasticity of demand	0.5
2. key parameters		
α	returns to scale on capital	0.6
ω	obsolescence costs	99%
κ	fixed adjustment costs	50%
ϕ	entry rate/competitive pressures	2
τ_S	informativeness of experiments	1/5

edge about the new technology—and of physical capital—there are fewer firms to compete profits with. The smooth-pasting condition for experimentation requires that the expected firm value upon experimentation and the firm value in the inaction regions meet tangentially at the investment thresholds $\underline{b}$ and $\overline{b}$, which define the experimentation region (the hashed blue area). The smooth-pasting condition for exploration further requires that the firm value lands flat at the edges of the domain defined by the exploration thresholds $\underline{a}$ and $\overline{a}$.

Figure 3 summarizes the exploration and experimentation strategy based on a sample path of the stock of knowledge, Z. As knowledge accumulates away from its reset point, it becomes optimal to explore a new technology (red arrows), either because the current technology is likely poor, or because new entrants erode profits away. The left-hand panel in Figure 4 shows the distribution of the length of a typical exploration cycle (i.e., the time spent by the firm operating a given technology; see Appendix E for derivations of this distribution) in the benchmark model (dashed line) and with experimentation (solid line). Clearly, experimentation shortens the expected length of the exploration cycle by allowing the firm to learn faster about the quality of its technology.

Upon entering the experimentation zone (the shaded blue area in Figure 3), the firm
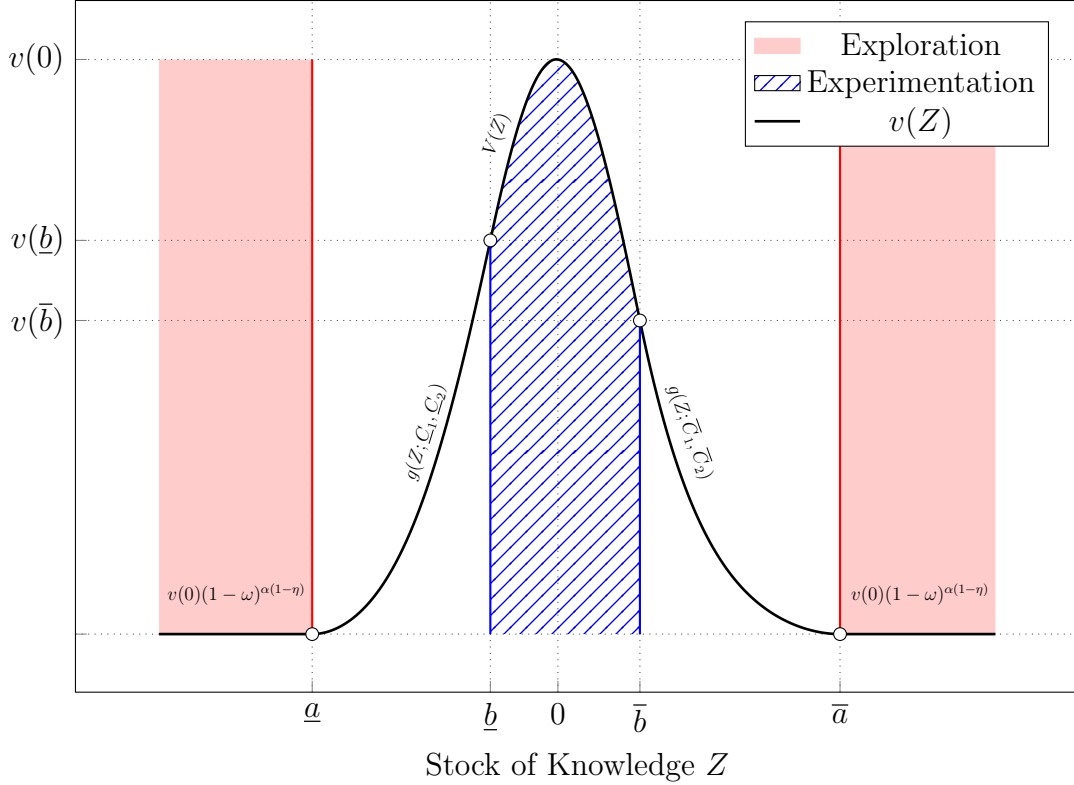
Figure 2: Firm value as a function of knowledge. The figure plots the firm value (solid black line) as a function of knowledge, Z, about the technology it operates. The blue hased area represents the experimentation regions, defined by the two thresholds, $\underline{b}$ and $\bar{b}$. The red shaded areas corresponds to the two exploration regions, each defined by the thresholds $\underline{a}$ and $\bar{a}$, respectively. The piecewise representation of the firm value in Eq. (21) is reported below the curve within each region.

goes through possibly multiple rounds of experimentation, investing in lumps and thereby reaping the high revenue product on capital and knowledge. Because experimentation outcomes are uncertain, each round of experiment affects knowledge differently, although on average in the same direction. Experimentation eventually causes knowledge to accumulate away from the experimentation zone (the red dots), either leading to immediate experimentation, or to inaction. The right-hand panel in Figure 4 shows the distribution of a typical experimentation round (i.e., the number of successive trials until first exit of $\mathscr{B}$). Under the baseline calibration, after exploring a new technology it most frequently takes 3 rounds of experimentation until the firm becomes either inactive or explores again.
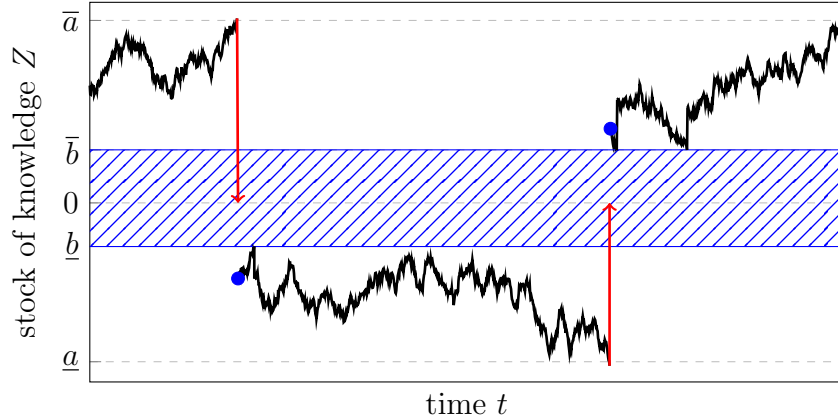
Figure 3: Sample path of Knowledge. The figure plots a sample path of the firm stock of knowledge (solid black line) over time. Blue dots correspond to times at which the firm chooses to experiment; red arrows correspond to times when the firm chooses to explore.

4 Investment, Tobin's q, and knowledge cycles

This section presents the testable predictions of the model. In particular, we want to understand how knowledge creation (the knowledge channel of Definition 1) affects investment and its relation with Tobin's q. The three main findings are that a firm invests in ways opposite to neoclassical predictions in the presence of the knowledge channel; experimentation risk arises as an origin of q mismeasurement (e.g., Erickson and Whited, 2000); the strength of the investment$-q$ relation concentrates at early to intermediate stages of the knowledge cycle depending on competitive pressures and experimentation risk. Throughout the section we refer to "experimentation risk" as the noise, ϵ, in the experimentation signal in Eq. (3).

We first introduce the appropriate definition of marginal q in this model, which is based on Abel and Blanchard (1986).

Definition 4. *(Marginal Product of Capital) Marginal q is the expected present value of a marginal unit of investment (Abel and Blanchard, 1986) scaled by relative profits, Π/K:*

$$q(Z) \equiv \int_{\mathbb{R}} \frac{V_K(\cdot, K + I, x)}{\Pi(\cdot)/K} \varphi(x; i(Z), Z)\, dx.$$
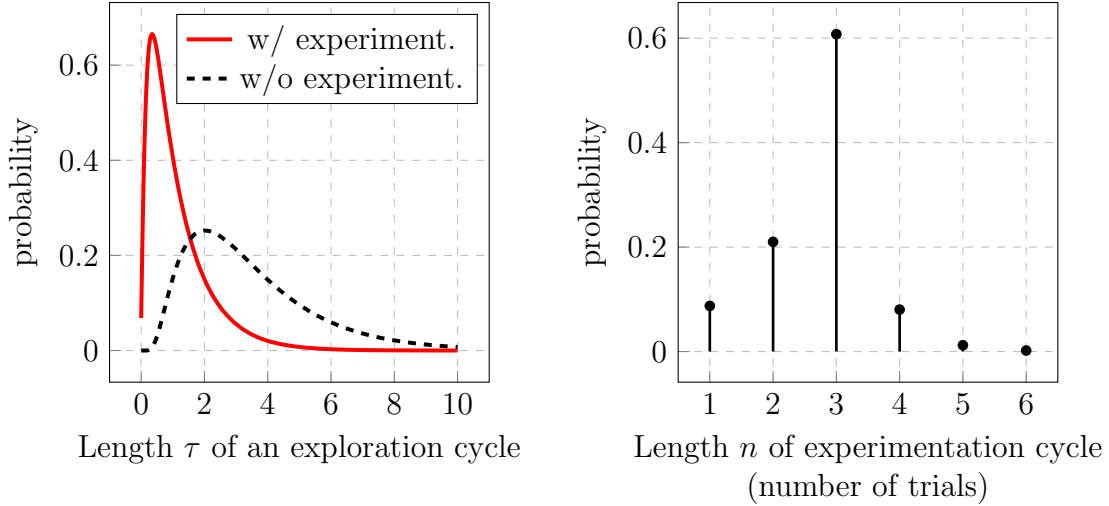
Figure 4: Distribution of the length of exploration and experimentation cycles. The left-hand panel plots the distribution of the length of an exploration cycle with (solid line) and without (dashed line) experimentation; derivations are reported in Appendix E. The right-hand panel shows the distribution of the length of an experimentation cycle.

This definition can be equivalently expressed in terms of intensive firm value as:

$$q(Z) = \alpha(1-\eta)(1+i(Z))^{\alpha(1-\eta)-1} \int_{\mathbb{R}} v(x)\varphi(x;i(Z),Z)\mathrm{d}x.$$

Note first that a (cosmetic) difference relative to Abel and Blanchard (1986) is on the relation between intensive firm value and the marginal product of capital. In their model and, more broadly, in the neoclassical investment framework (e.g., Hayashi, 1982), firm value is often assumed to be homogeneous in K so that the intensive firm value represents marginal q. Our specification of adjustment costs, however, implies that firm value is homogeneous in profits, Π, as opposed to K. It is thus natural to scale firm value by relative profits, Π/K, so that q remains a sufficient statistic for investment in the benchmark model, consistent with the neoclassical insight. We then study how the knowledge channel affects this insight (Section 4.1).

Second, and more importantly, since the effect of investment on knowledge is uncertain, marginal q is the value of an additional unit of capital *averaged* over all possible experi-

mentation outcomes.[12] An important consequence of taking this average is that the econometrician cannot compute q so defined, because she cannot control for all experimentation outcomes that are possible *ex-ante*. Rather, she can control for experimentation outcomes *ex-post*, which leads to q mismeasurement. In this context, it is important to understand how measured q relates to actual q (Definition 4), how they covary with the knowledge channel (Section 4.2), and how the relation between measured q and investment varies over the knowledge cycle (Section 4.3), making the knowledge cycle a possible instrument for mismeasurement.

4.1   Knowledge and Corporate Investment

We start from the observation that, absent the knowledge channel (the benchmark model of Section 3.1), investment is determined through the traditional neoclassical trade-off. In the absence of experimentation, the effect of investment on knowledge is known, so that q in Definition 4 is known ex-ante (i.e., it does not involve taking an expectation):

$$q(Z) \equiv \alpha(1 - \eta)v(Z).$$

Substituting this definition in the first-order condition for optimal investment in Eq. (11) thus delivers the traditional investment$-q$ relation:

$$i(Z) = \frac{1}{\gamma}q(Z), \tag{24}$$

which states that *(a)* marginal q is a sufficient statistic for investment and that *(b)* the sensitivity of investment to q is constant ($\equiv 1/\gamma$) throughout the knowledge cycle.

To determine how the knowledge channel affects the neoclassical relation, we use the first-order condition for optimal investment, i, from the Bellman Eq. (22). Substituting the definition of the knowledge channel (Definition 1) and of marginal q (Definition 4) yields:

$$0 = \underbrace{q(Z) - \gamma i(Z)}_{\text{neoclassical trade-off}} + \underbrace{c(Z)}_{\substack{\text{knowledge} \\ \text{channel}}}. \tag{25}$$

---

[12] A similar definition prevails when investment becomes productive with a one-period delay (e.g., Caballero and Leahy, 1996).

This equation determines a knowledge-contingent investment plan, $i(Z)$, for all values of knowledge within the experimentation band, $Z \in \mathscr{B}$. The first channel driving investment in Eq. (25) is the neoclassical channel, as present in the benchmark model (see Eq. (24)). Absent experimentation benefits (i.e., $\tau_S \equiv 0$), marginal revenue product of capital equals marginal adjustment costs (Hayashi, 1982). The second channel driving investment is the knowledge channel, which captures how investment affects firm value by improving knowledge. In Figure 5 we compare the relation between investment and knowledge as determined from Eq. (25) (right panel) to its counterpart in the benchmark model (left panel).
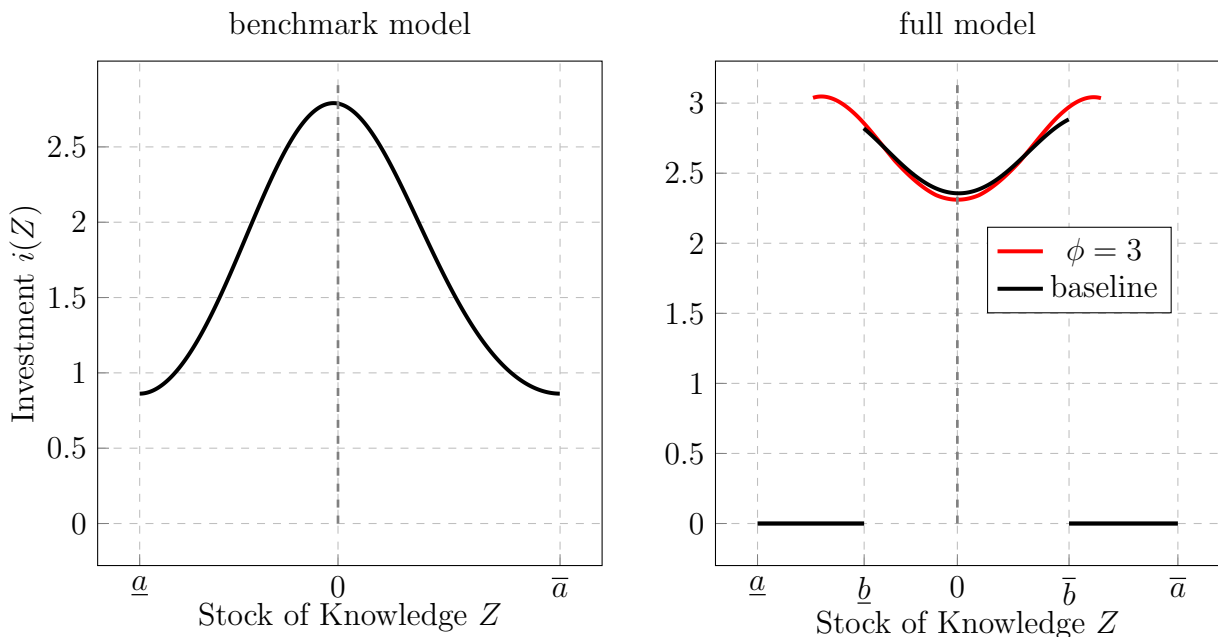

Figure 5: Knowledge-contingent investment plan. This figure plots investment as a function of knowledge, Z, in the benchmark model of Section 3.1 (left panel) and in the full model of Section 3.2 (right panel). In the right-hand panel, the black curve represents investment under baseline parameter values; the red curve represents investment under high competitive pressures ($\phi = 3 > 2$). Baseline parameter values are defined in Table 1.

The knowledge channel turns the neoclassical relation between investment and knowledge on its head. Because the neoclassical relation holds in the benchmark model, investment is proportional to q: the left panel is a scaled version of Figure 1. Consistent with the neoclassical insight, investment is highest when marginal q is highest following recent exploration, and weakest when marginal q is weakest as exploration becomes imminent. Thus, in the

benchmark model, the neoclassical insight predicts a decreasing relation between investment and knowledge, $|Z|$. In the presence of the knowledge channel (the right panel), as in the benchmark case, investment occurs following recent exploration when investment benefits are high. In other words, marginal q determines the location of the experimentation band. Within the experimentation band, however, investment is weakest following exploration, and strengthens as knowledge accumulates. The knowledge channel thus implies an increasing relation between investment and knowledge, in contrast to the neoclassical insight.

An alternative interpretation of Figure 5 is that an outside observer who reads the right-hand panel based on neoclassical insights (that is, omitting the knowledge channel) concludes that the firm under-invests following exploration and over-invests as knowledge accumulates. Interestingly, increased competitive threats (the red line) stimulates this effect, rather than mitigating it.[13] A common view (e.g., Gutierrez and Philippon (2017)) is that increased competition should reduce under-investment when valuation ratios are high. In this model, entry threats may or may not be consistent with this view depending on the stage of the knowledge cycle.

To understand how the knowledge channel overturns neoclassical insights and how increased entry threats may lead to (perceived) under-investment, we use Stein's lemma to decompose the knowledge channel in Eq. (8) as:[14]

$$c(Z) = (1+i)^{\alpha(1-\eta)} \frac{\tau_S}{2} \Big(\underbrace{\frac{Z}{\sqrt{1+\tau_S i}} \int_{\mathbb{R}} v'(x)\varphi(x;i(Z),Z)\mathrm{d}x}_{\text{marginal product of knowledge}} + \underbrace{\int_{\mathbb{R}} v''(x)\varphi(x;i(Z),Z)\mathrm{d}x}_{\text{attitude towards experimentation risk}} \quad (26)$$

The first term represents the marginal revenue product of knowledge, which captures the expected value of a marginal unit of knowledge. Was firm value linear in knowledge, Bayes' rule (see Eq. (17)) would imply that investing 1% of capital in knowledge increases the firm value with expected returns-to-scale, $\approx \tau_S/2$. But, because firm value is nonlinear, the firm's attitude towards experimentation risk affects returns-to-scale on knowledge; this effect is measured by the second term in Eq. (26).

That the firm's attitude towards experimentation risk matters for its investment decision is perhaps surprising if one considers that the firm is assumed to be risk-neutral. In partic-

---

[13]Reducing experimentation noise has a similar effect.

[14]The resulting relation only makes sense if one ignores the points at which the firm value is not twice continuously differentiable.

ular, the firm is risk-neutral towards profit risk, an assumption that underlies the *ex-ante* representation of firm value in Eq. (6). However, experimentation risk affects firm value *ex-post*. As a result, depending on the curvature of firm value (as measured by v'' in Eq. (26)), experimentation is either a gamble or a risk towards which the firm is averse. Since experimentation occurs over a region of knowledge where firm value is concave, the firm is endogenously "information averse" (Andries and Haddad, 2019).

In light of the decomposition in Eq. (26), what Figure 5 actually shows is that the firm's attitude towards experimentation risk dictates its investment decision, as opposed to q or marginal product of knowledge. Note first that marginal product of knowledge in Eq. (26) works hand in hand with marginal q, thus reinforcing the neoclassical channel. Following recent exploration (when the firm has little knowledge about the new technology it has selected), the marginal product of knowledge is high; conversely, when the technology is likely of high (or low) quality, and new entrants make exploration imminent, the marginal product of knowledge is low. However, following exploration the firm value is concave, the firm thus dislikes experimentation risk, and invests by "tatonnement" despite high benefits to investment. As competitive pressures rise (exploration is imminent), the firm value becomes convex, and the firm is increasingly willing to take bets—experimentation becomes a "gamble" on future exploration. The main conclusion is that the firm's attitude towards risk acts as a dominating force that overturns the neoclassical channel.

The next question we ask is, how does an empiricist perceive the relation between investment and knowledge through the lens of q theory. To answer this question, we must first determine how an empiricist measures q, a matter we now turn to.

4.2   Experimentation risk as an origin of q mismeasurement

The knowledge channel creates a gap (see Eq. (25)) between marginal q and investment—modern q theory fails in this model. Actually, investment and q are either unrelated (outside the experimentation region) or negatively related (within the experimentation region). Specifically, suppose the econometrician could run the regression:

$$i(Z_t) = a + bq(Z_t) + \epsilon(Z_t). \tag{27}$$

She would then obtain the following estimate for the investment$-q$ sensitivity:

$$b = \underbrace{1/\gamma}_{\substack{\text{neoclassical} \\ \text{sensitivity}}} \bigg(\underbrace{\frac{\text{cov}\left(q(Z)\mathbf{1}_{Z \in \mathscr{B}}, q(Z)\right)}{\mathbb{V}(q(Z))}}_{\equiv \beta_q} + \underbrace{\frac{\text{cov}\left(c(Z)\mathbf{1}_{Z \in \mathscr{B}}, q(Z)\right)}{\mathbb{V}(q(Z))}}_{\text{knowledge-channel gap, } \beta_c} \bigg). \qquad (28)$$

As we demonstrate numerically below, the sign of each of these covariances is ambiguous, and depends on the extent of entry threats. However, since we have shown that the firm's risk attitude creates a negative relation between investment and q, we know that the sign of these covariances is always such that b is negative. As a result, the econometrician consistently (and correctly) rejects q as a sufficient statistic for investment.

A complication, however, is that experimentation risk makes both the knowledge channel and marginal q unobservable to the econometrician. The econometrician's conclusions are thus affected by the extent to which she mismeasures q (e.g., Erickson and Whited, 2000), and by how measured q covaries with the knowledge channel. In particular, we make the following assumption as to what the econometrician observes.

Assumption 1. *The econometrician can instrument for the firm's stock of knowledge, but cannot control for experimentation risk.*

The first part of the assumption allows us to subsequently study how investment$-q$ sensitivity varies over the knowledge cycle; the second part means that the empiricist cannot compute the expectation involved in the definition of q (and in the knowledge channel).

Another, long-standing complication is that marginal q (even *ex-post*) is difficult to measure empirically. Empiricists instead use *average Q*, which also proves to do better than marginal q as a right-hand side variable in an investment regression in the presence of fixed adjustment costs (e.g., Caballero and Leahy (1996)). Given our assumption that adjustment costs are proportional to revenue, average Q in this model is defined as a price-revenue ratio:

$$Q(Z) \equiv \frac{V(\cdot, K+I, Z)}{\Pi(\cdot, K)} = (1 + i(Z))^{\alpha(1-\eta)} v(Z). \qquad (29)$$

Furthermore, because the empiricist cannot control for all experimentation outcomes that are possible *ex-ante* (Assumption 1), this definition of average Q is measured *ex-post*.

Suppose now that the empiricist runs the regression in Eq. (27), but using average Q as a right-hand side variable. Her estimate then depends on how (observed) average Q covaries

with the (unobserved) marginal q and with the (unobserved) knowledge channel:

$$\widehat{b} = \frac{1}{\gamma} \underbrace{\frac{\mathrm{cov}\left(q(Z)\mathbf{1}_{Z \in \mathscr{B}}, Q(Z)\right)}{\mathbb{V}(Q(Z))}}_{\equiv \widehat{\beta}_q} + \frac{1}{\gamma} \underbrace{\frac{\mathrm{cov}\left(c(Z)\mathbf{1}_{Z \in \mathscr{B}}, Q(Z)\right)}{\mathbb{V}(Q(Z))}}_{\equiv \widehat{\beta}_c}. \tag{30}$$

Whether average Q is a good statistic for investment in this model depends on its relation to marginal q and the knowledge channel. Because marginal q is a poor statistic for investment in the model, we expect $\widehat{\beta}_q$ to be weak, and $\widehat{\beta}_c$ to be substantially weaker than β_c. In Figure 6 we proceed numerically and plot these coefficients along with the sensitivity estimate $\widehat{b}$ against their values β_q, β_c and b had we used marginal q as a right-hand side variable; we do so for different values of competitive pressure, ϕ.
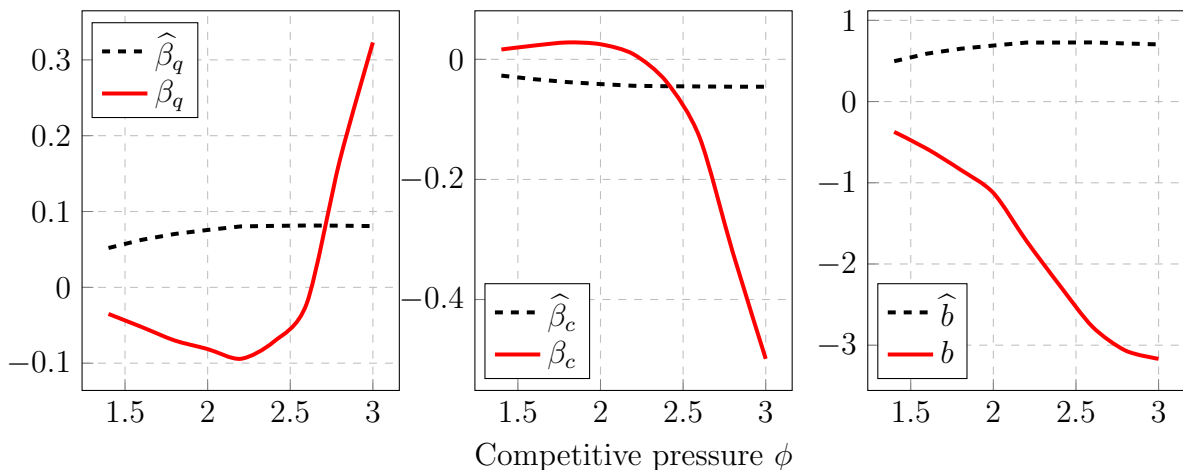

Figure 6: Investment$-q$ sensitivity as a function of competitive pressures. The figure plots the coefficients β_q and $\widehat{\beta}_q$ (left panel), β_c and $\widehat{\beta}_c$ (middle panel), and b and $\widehat{b}$ (right panel) as a function of competitive pressure, ϕ; these coefficients are defined in Eqs. (28)–(30).

We start by describing the coefficients β_q (left panel) and β_c (middle panel), and the resulting sensitivity estimate b (right panel) using marginal q as a right-hand variable. On the one hand, we expect the knowledge channel to covary negatively with marginal q within the experimentation band. On the other hand, the experimentation band coincides with high values of q on average, which should lead the knowledge channel to covary positively with marginal q unconditionally. The middle panel shows that the latter effect dominates for the lower range of competitive pressure, whereas the former effect dominates as compe-

tition intensifies. Furthermore, absent the knowledge channel, under pure fixed costs and irreversibility, investment and marginal q are negatively related (e.g., Caballero and Leahy (1996)). Intuitively, when investment occurs $q = 0$, and $q > 0$ when it does not. It follows that $\mathbb{E}[qI] = 0$, and thus $\mathrm{cov}(q, I) = -\mathbb{E}[q]\mathbb{E}[I]$, which is negative under irreversibility, $I \geq 0$. Adding a convex cost can make this covariance positive. The left panel shows that for low entry threats the former effect dominates, whereas for high threats the latter effect dominates. On balance the net effect is that investment and marginal q are negatively related (the right panel), the more so the higher competitive threat is.

Using *ex-post* average Q as a right-hand variable produces a strong attenuation bias (dashed lines are of smaller magnitude relative to solid lines). Unlike covariances involving marginal q, the sign of covariances involving average Q is unambiguous. Average Q always covaries positively with marginal q, and negatively with the knowledge channel. On net the former covariance dominates leading to a positive measured sensitivity of investment to average Q, which strengthens as competitive pressures rise. The point is that *ex-post* average Q performs much better than marginal q as a statistic for investment.

We have established that *ex-post* average Q is a reasonable measure of investment in this model; we now wish to identify at which stages of the knowledge cycle its statistical power concentrates.

4.3   The investment$-q$ relation over the knowledge cycle

To evaluate the $q-$investment relation at different stages of the knowledge cycle, we proceed as in empirical studies and use the model as a laboratory. We simulate time series of *ex-post* average Q (as defined in Eq. (29)) and investment, which we illustrate in Figure 7. Due to fixed adjustment costs, investment occurs infrequently and in lumps, the size of which varies with the firm's stock of knowledge. Furthermore, because investment occurs within the experimentation range, it also clusters in time. Investment clustering is a feature that is specific to our premise that knowledge is a by-product of economic activity. Because average Q measures firm value even when the firm does not invest, it moves all the time.

We then perform the regression in Eq. (27) based on average Q within each quintile k of
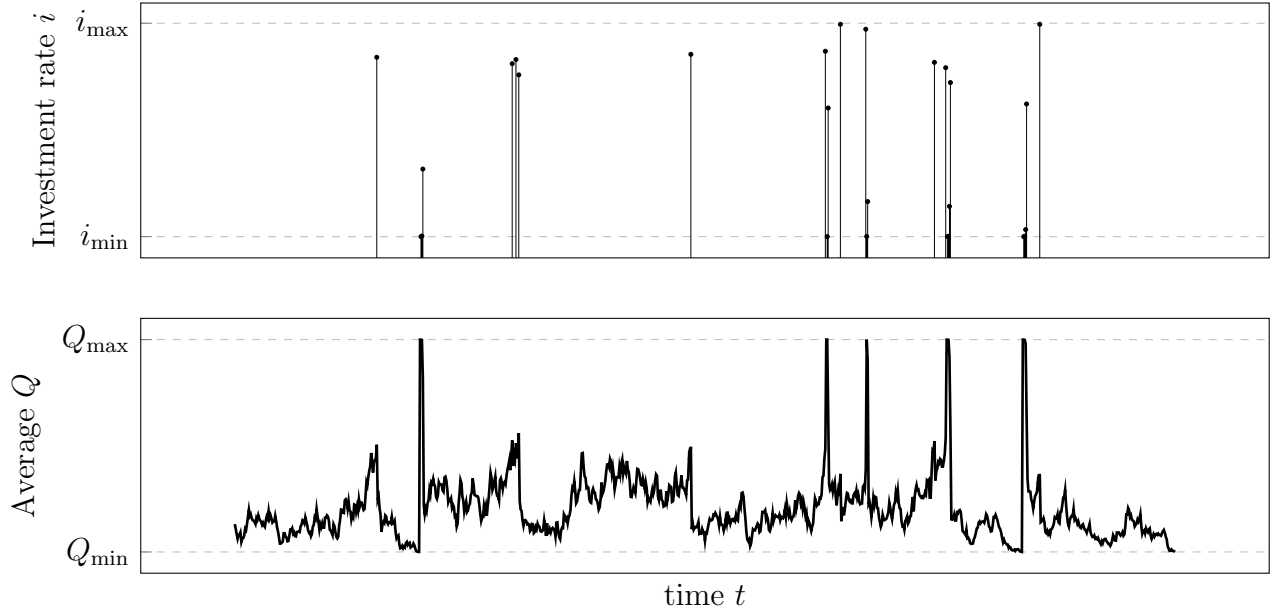
Figure 7: Sample path of investment and average Q. The upper and lower panel plot a sample path of the investment rate and the associated average Q, respectively, over time. The upper panel shows that investment occurs in lumps and clusters.

knowledge $|Z|$ according to:

$$i(Z_t) = a_k + b_k Q(Z_t) + \epsilon_t, \quad |Z_t| \in \text{quintile}_k, \quad k = 1, ..., 10. \tag{31}$$

In Figure 8 we plot the resulting investment$-q$ sensitivity b_k for each quintile k under baseline parameter values (middle panel), low experimentation noise (left panel), and high competitive pressures (right panel). As a benchmark against which to evaluate this figure, recall that in the benchmark model investment$-q$ sensitivity, $b_k \equiv 1/\gamma > 0$, is fixed throughout the knowledge cycle and thus does not vary across quintiles.

The presence of fixed adjustment costs in the model creates inaction regions over which investment is mechanically unrelated to Q. In each panel of Figure 8, this outcome is associated with the higher quintiles of knowledge within which investing is suboptimal (i.e., whenever $Z \notin \mathscr{B}$). In contrast, over lower quintiles a positive relation between investment and average Q arises. The mechanism of the previous section dictates the sign of this relation—average Q covaries weakly with both marginal q, which is a poor proxy for

investment, and the knowledge channel, which interacts negatively with investment. Hence, unlike in the benchmark model, the relation between average Q and investment varies over the knowledge cycle and its statistical power concentrates in lower quintiles of knowledge.

Interestingly, the statistical power of average Q concentrates in lower to intermediate quintiles depending on competitive pressures or the extent of experimentation noise. Raising competitive pressures or reducing experimentation noise (the left and right panels) affects investment$-Q$ sensitivity in similar ways. Because investment becomes more aggressive in both cases, the experimentation band widens and the sensitivity of investment to Q now concentrates in intermediates quintiles of knowledge. Raising competitive pressures and reducing experimentation noise further increases the magnitude of the sensitivity.
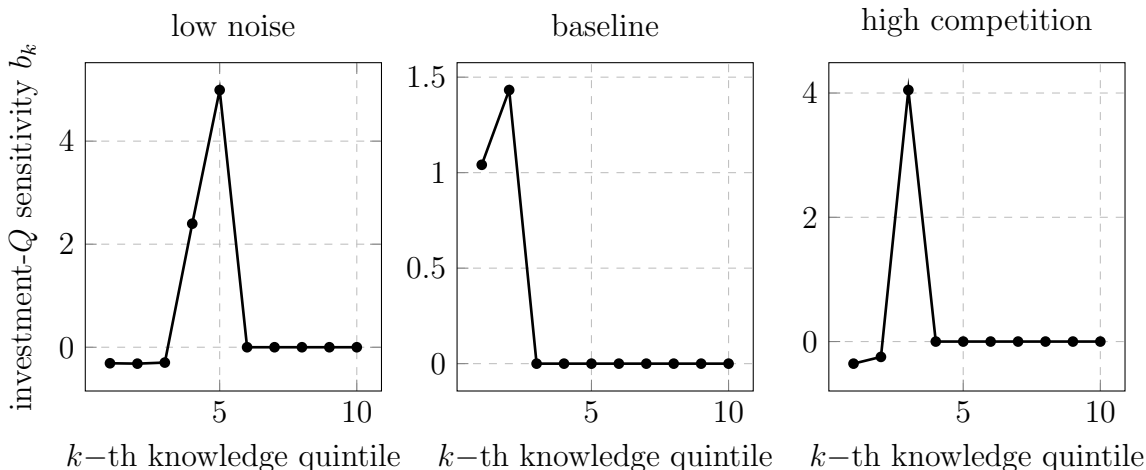
Figure 8: Investment$-q$ sensitivity as a function of knowledge quintiles. Each panel plots the sensitivity of investment to Tobin's q (the estimated slope coefficient in Eq. (31)) within quintiles of the (absolute value of the) stock of knowledge, $|Z|$. The middle panel presents this relation under baseline parameter values (Table 1), the left panel under low experimentation noise ($\tau_S = 0.15$), and the right-hand panel under high competitive pressures ($\phi = 3$).

5   Illustrative Empirical Evidence

Testing the implications of the model is challenging as it requires the ability to measure firms' position on their knowledge cycle (i.e., Z). To assess the qualitative relevance of the knowledge channel in the data, we propose to use the new text-based measure of product life cycles recently developed by Hoberg and Maksimovic (2019) as an empirical proxy for Z, and examine how Z interacts with physical investment and Q in a large sample of publicly-listed firms.

5.1   Data and Measurement

Following the characterization of the life cycle of firms' products introduced by Abernathy and Utterback (1978), Hoberg and Maksimovic (2019) use a textual analysis of firms' 10-K reports to identify the relative position of each firm (and year) in this cycle. We strictly follow the methodology of Hoberg and Maksimovic (2019) and consider that product life cycle follows four distinct stages: (1) product innovation, (2) process innovation, (3) maturity, and (4) decline. We recognize that product and knowledge cycles may not perfectly overlap in reality. Thus our proxy is arguably imperfect. Yet, we posit that both cycles are likely to be positively correlated for most firms. On that ground, we conjecture that firms' stock of knowledge is low (i.e., Z is close to zero) at the beginning of the product cycle (i.e., in the product innovation stage), and increases (i.e., Z moves away from zero) in later stages (i.e., in process innovation).

We consider all firm-years present in the universe of Compustat between 1996 and 2016 with adequate 10-K data using the WRDS SEC Analytics package. We determine the intensity with which a given firm is in each stage in a given year using an "anchor-phrase" method to determine the number of paragraphs in its 10-K with words contained in four distinct (pre-determined) lists associated with each stages. We then define the four life cycles variables by dividing each of the four individual paragraph counts by the total paragraphs counts for the four. We refer to these four variables as Life1, Life2, Life3, and Life4, respectively.[15] For example, a firm scores highly on Life1 if it intensively discusses issues related to product or service innovation or development in its 10-K. We exclude observations with

---

[15]By constructoin, Life1+Life2+Life3+Life4=1 for each firm-year.

missing data on capital expenditures, sales and assets below $1 million. Our sample includes 63,199 firm-year observations.

5.2   Investment and Q across the Knowledge Cycle

First, to examine the links between physical investment, Q, and the stages of firms' product life cycle, we replicate the empirical analysis of Hoberg and Maksimovic (2019). As discussed above (Section 4.1), absent the knowledge channel, the model predicts that both investment and Q (i.e, average Q) are *decreasing* in knowledge (Z). Indeed, in the (neoclassical) benchmark, investment is the highest when Q is the highest following recent exploration (when knowledge is reset to zero). In contrast, in the presence of the knowledge channel, investment is *increasing* in knowledge (due to endogenous information aversion), but Q continues to be *decreasing* in knowledge (see Figure 5).

To assess which pattern holds in the data, we follow existing research and define investment as capital expenditures scaled by beginning of the period assets, and Q as market value of assets divided by their book value (e.g., Gutierrez and Philippon (2017)). We then regress investment and Q on the four product life cycle variables (with and without year fixed effects).[16] Figure 9 displays the estimated coefficients (together with their 95% confidence bounds). In the left panel, we observe that firms' physical investment is about two times lower at the beginning of the cycle (i.e., when knowledge is low) than in the second stage, when more knowledge has accumulated. However, the right panel indicates that Q is more than three times larger in the first stage of the cycle than in the second stage.[17] This opposite patterns between investment and Q over the initial stages of the knowledge cycle is consistent with the predicted role of knowledge in the model.

Next, we explore how investment correlates with Q along the knowledge cycle. A key insight from our analysis is that, in the benchmark model, the investment-Q relation remains fixed over the knowledge cycle, but in the presence of the knowledge channel it does not, as illustrated in Figure 8. To assess whether and how the correlation between investment and Q changes over the knowledge cycle, we simply regress investment on the four life

---

[16]We standardize the four life cycles variables by their sample standard deviation to facilitate the economic interpretation of the estimated coefficients, and we cluster standard errors at the firm level.

[17]Note that these results are consistent with the overall findings reported in Hoberg and Maksimovic (2019)
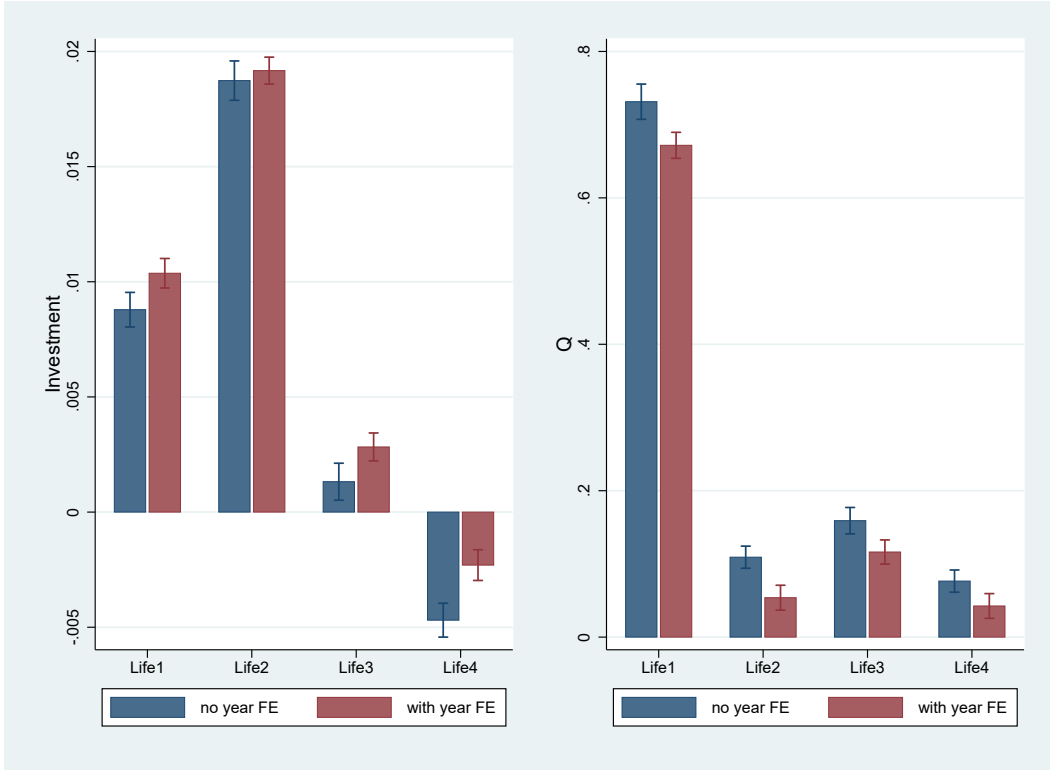
Figure 9: Investment and Q over the product life cycle. The left panel plots investment and the right panel plots average Q, both as a function of the four stages of the product cycle defined in Hoberg and Maksimovic (2019), the empirical procedure of whom we follow.

cycles stages variables and their interaction with Q (again with and without year fixed effects).[18] This approach closely resembles that used on the simulated data in Eq. (31), as the estimated coefficients on the four interaction variables tracks the empirical correlation between investment and Q across the four cycle stages.

In line with Hoberg and Maksimovic (2019) and in sharp contrast with the prediction of the benchmark model, Figure 10 indicates that the sensitivity of investment to Q varies considerably over the knowledge cycle. Consistent with the knowledge channel, investment appears to be negatively related to Q in the first stage of the knowledge cycle. In the model, when firms' explore new technologies and possess little knowledge about them, Q is high but investment is low since firms do not experiment yet. In later stages, however,

---

[18]We again standardize the interacted variables by their sample standard deviation to facilitate the economic interpretation of the estimated coefficients, and we cluster standard errors at the firm level.
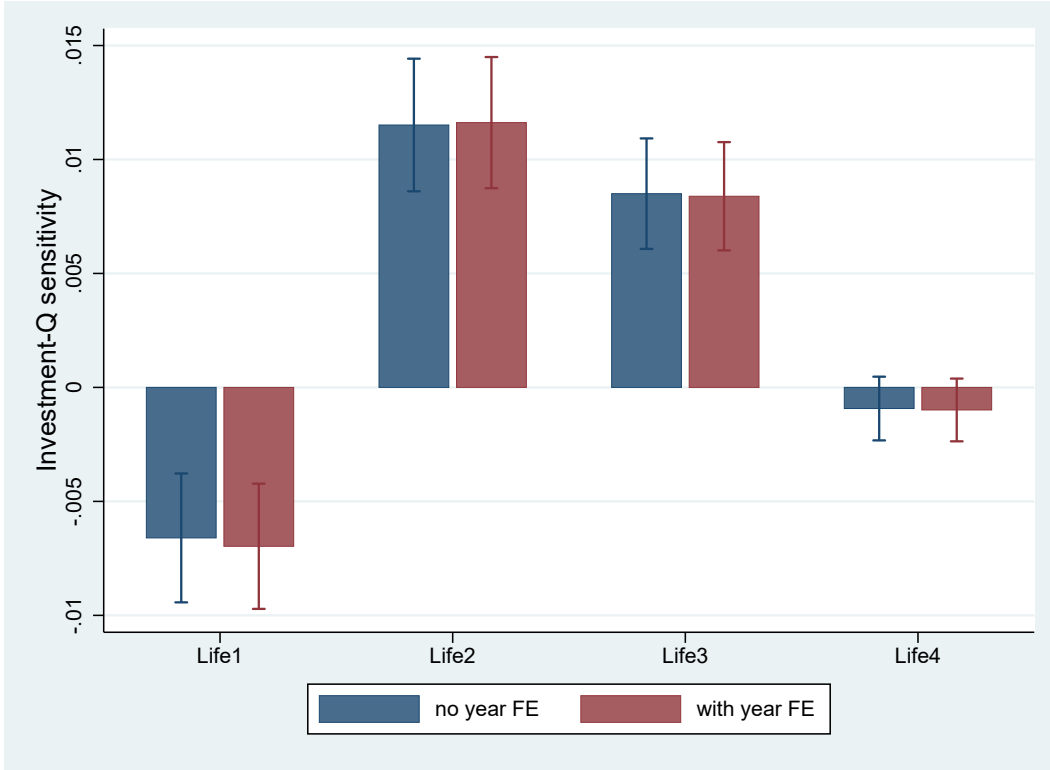
Figure 10: Investment$-Q$ sensitivity over the product life cycle. This figure plots the sensitivity of investment to Q as a function of four stages of the product cycle, as defined in Hoberg and Maksimovic (2019), and the empirical procedure of whom we follow.

the accumulated knowledge pushes firms to experiment by investing, generating stronger investment-Q sensitivity. This prediction is supported by Figure 10, which reveals a positive and strong correlation between investment and Q in the second and third stage. Taken at face value, and compared to the simulated results displayed in Figure 8, the patterns of Figure 10 suggest that the data appear consistent with calibrations of the model in which the noise associated with experimentation (τ_S) is low or competitive threats (ϕ) are high.

5.3   Aggregate Trends

Finally, we study whether the recent aggregate trends in firms' investment and Q could be in line with the model's qualitative insights. In particular, (Gutierrez and Philippon, 2017) and (Alexander and Eberly, 2018) document an aggregate decline in investment in the last twenty years, despite no noticeable decline in Q (and other fundamentals). Crouzet

and Eberly (2018) further report that the decline in physical investment is related to the increasing relevance of intangibles. Unsurprisingly (but reassuringly), these trends are also present in our sample. Figure 11 plots the average evolution of investment and Q (the left panel), and displays the evolution of the sensitivity of investment to Q (the right panel).[19]
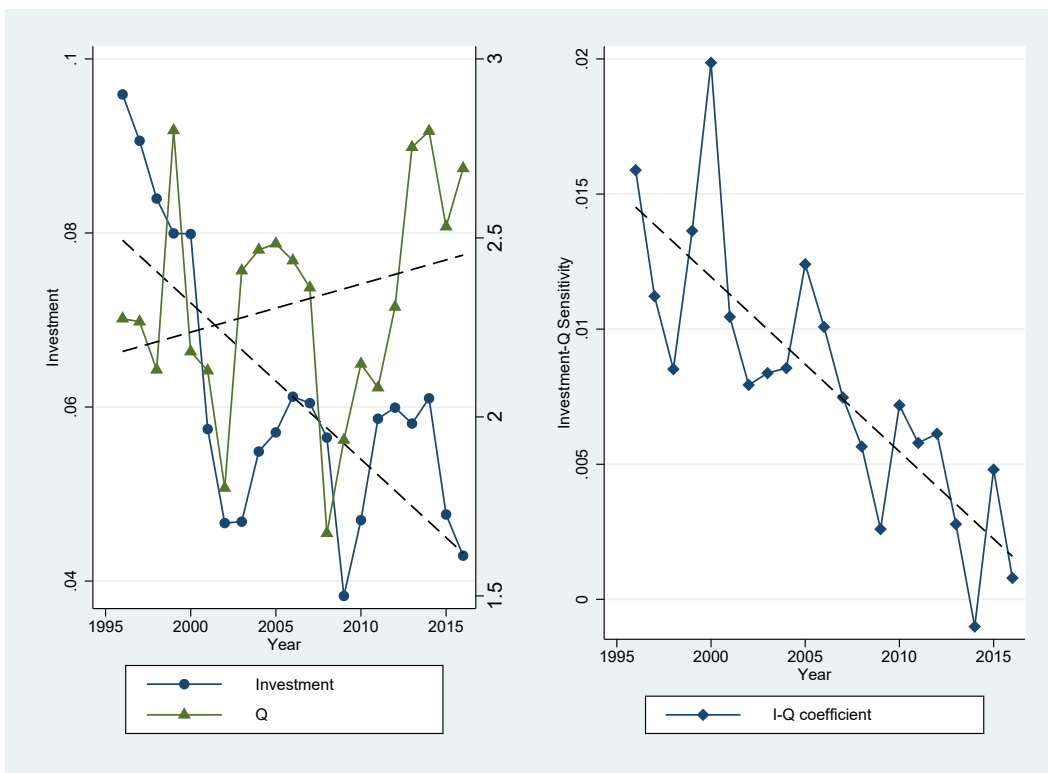
Figure 11: Q, investment, and its sensitivity to Q over time. The left panel plots the average evolution of investment and average Q over the sample period. The right panel plots the evolution of sensitivity of investment to Q over the sample period.

Interpreted in light of the model, the aggregate shifts in investment and Q could stem from an aggregate shortening of the average knowledge cycle. Specifically, weaker physical investment despite high Q could obtain if firms are exploring new technologies more frequently, thereby staying longer in the low knowledge region (i.e., Z close to zero). Such a rise in exploration intensity could happen for instance if high quality technologies (i.e., large values of M in the model) are getting harder to find as suggested by Bloom et al. (2017), pushing firms to explore more intensively. Alternatively, an increase in the cost and/or

---

[19]We obtain it by estimating annual investment regressions, and plotting the estimated coefficients.

risk associated with experimentation (i.e., higher noise τ_S) due to increased technological complexity could also induce a shortening of the knowledge cycle.[20]
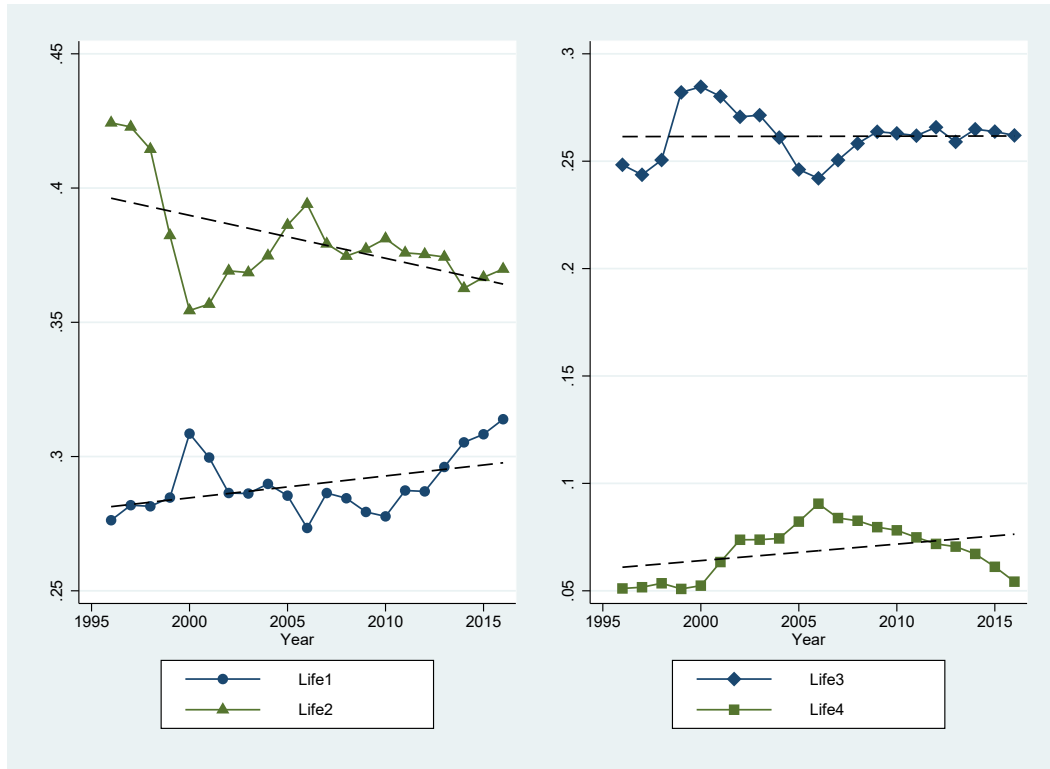
Figure 12: Evolution of the four life-cycle stages over time. The left panel plots the evolution of the first two stages over the sample period. The right panel plots the evolution of sensitivity of the last two stages stages over the sample period.

To assess whether weaker investment despite high Q in the recent period may be due to a shortening of the knowledge cycle inducing more exploration, we plot in Figure 12 the evolution of the four life-cycle stages over our sample period. The left panel displays the evolution of the first two stages, and the right panel focuses in the last two stages. Consistent with the rise in "business dynamism" noticed by Hoberg and Maksimovic (2019), the left panel unambiguously indicates a shortening of firms' knowledge cycles. The average value of Life1 is trending upwards, increasing from 27% in 1996 to 33% in 2016. In contrast, the average value of Life2 is decreasing from 43% to 36% over the last twenty years. The right

---

[20]In the model, other possibilities include heightened competitive threats or lower return to scale of the production technologies. Yet, they seem implausible given recent evidence documenting increases in market concentration, and the fact that the increasing digitization of the economy should augment returns to scale).

panel suggests no significant change in the average value of Life3, and a marginal increase in Life4. Overall, the trends displayed in Figure 12 suggest that on average firms are spending relatively more time in the exploration stage, where physical investment is predicted to be weak, and Q to be high. While we acknowledge that our empirical analysis remains qualitative and cursory, it broadly supports the predictions of the model and the important role of the knowledge channel in explaining corporate investment.

6   Conclusion

This paper presents a model of how knowledge is created within firms, and how this process of knowledge creation affects firms' investment decisions. A key assumption is to view firms as storehouses of information, and knowledge as a by-product of risky experimentation. The trade-off between exploration of new technologies and experimentation witht the current technology creates endogenous knowledge cycles. Experimentation, which acts as a "knowledge channel" and as an origin of Q mismeasurement, makes firm endogenously risk averse. This knowledge channel causes investment to be increasing in knowledge, its relation to Q to vary over the knowledge cycle, and to concentrate at early stages of the cycle. These patterns, which competition exacerbates, are in sharp contrast to neoclassical predictions that abstract away from the knowledge channel. We use a new text-based measure of product life cycle to proxy for the knowledge cycle and report evidence of the empirical relevance of the knowledge channel. We argue that the knowledge channel could explain abnormally low investment since the early 2000's, as a result firms spending more time in earlier stages of their knowledge cycle during this period.

This paper suggests that competition may have ambiguous effects on investment depending on the stage of the knowledge cycle—firms may appear to under-invest at earlier stages and to over-invest at later stages. In future research we wish to examine how these competitive forces play out in the context of an industry equilibrium.

References

Abel, A. and J. Eberly (1994). A unified model of investment under uncertainty. *American Economic Review 84*(5), 1369–84.

Abel, A. B. and O. J. Blanchard (1986). The present value of profits and cyclical movements in investment. *Econometrica 54*(2), 249–273.

Abernathy, W. and J. Utterback (1978). Patterns of industrial innovation. *The RAND Journal of Economics 80*, 41–47.

Aghion, P., P. Bolton, C. Harris, and B. Jullien (1991). Optimal learning by experimentation. *The Review of Economic Studies 58*(4), 621–654.

Alexander, L. and J. Eberly (2018). Investment hollowing out. *IMF Economic Review* (66), 5–30.

Andrei, D., W. Mann, and N. Moyen (2018). Why did the q theory of investment start working? *The Journal of Financial Economics*.

Andries, M. and V. Haddad (2019, Oct). Information Aversion. NBER Working Papers 23958, National Bureau of Economic Research, Inc.

Arrow, K. (1962). The economic implications of learning by doing. *Review of Economic Studies 29*(3), 155–173.

Balsmeier, B., L. Fleming, and G. Manso (2011). Motivating innovation. *The Journal of Finance 66*(5), 1823–1860.

Benkard, C. L. (2000, September). Learning and forgetting: The dynamics of aircraft production. *American Economic Review 90*(4), 1034–1054.

Bergemann, D. and U. Hege (2005). The financing of innovation: Learning and stopping. *The RAND Journal of Economics 36*(4), 719–752.

Berk, J. B., R. C. Green, and V. Naik (2004, 01). Valuation and Return Dynamics of New Ventures. *The Review of Financial Studies 17*(1), 1–35.

Bloom, N., C. Jones, J. van Reenen, and M. Webb (2017). Are ideas getting harder to get? NBER Working Paper 23782.

Bloom, N., R. Sadun, and J. van Reenen (2016). Management as a technology? *Working Paper*.

Bresnahan, T., S. Greenstein, D. Brownstone, and K. Flamm (1996). Technical progress and co-invention in computing and in the uses of computers. *Brookings Papers on Economic Activity* (1996), 1–83.

Caballero, R. J. and J. V. Leahy (1996). Fixed costs: The demise of marginal q. *Unpublished Manuscript*.

Caballero, R. J. and R. S. Pindyck (1996, August). Uncertainty, Investment, and Industry Evolution. *International Economic Review 37*(3), 641–662.

Callander, S. (2011, October). Searching and learning by trial and error. *American Economic Review 101*(6), 2277–2308.

Cooper, I. (2006). Asset pricing implications of nonconvex adjustment costs and irreversibility of investment. *The Journal of Finance 61*(1), 139–170.

Corrado, C. A. and C. R. Hulten (2010). How Do You Measure a "Technological Revolution"? *American Economic Review 100*, 99–104.

Crouzet, N. and J. Eberly (2018, May). Intangibles, investment, and efficiency. *AEA Papers and Proceedings 108*, 426–31.

Eisfeldt, A. L. and D. Papanikolau (2013). Organization Capital and the Cross-Section of Expected Returns. *Journal of Finance 68*, 1365–1406.

Erickson, T. and T. M. Whited (2000). Measurement error and the relationship between investmentand ¡em¿q¡/em¿. *Journal of Political Economy 108*(5), 1027–1057.

Ericson, R. and A. Pakes (1995). Markov-perfect industry dynamics: A framework for empirical work. *The Review of Economic Studies 62*(1), 53–82.

Farboodi, M., R. Mihet, T. Philippon, and L. Veldkamp (2019, May). Big data and firm dynamics. *AEA Papers and Proceedings 109*, 38–42.

Grossman, S. J., R. E. Kihlstrom, and L. J. Mirman (1977). A bayesian approach to the production of information and learning by doing. *The Review of Economic Studies 44*(3), 533–547.

Gutierrez, G. and T. Philippon (2017). Investmentless growth: An empirical investigation. *Brookings Papers on Economic Activity*, 89–169.

Hackbarth, D. and T. Johnson (2015, 05). Real Options and Risk Dynamics. *The Review of Economic Studies 82*(4), 1449–1482.

Hayashi, F. (1982). Tobin's Marginal q and Average q: A Neoclassical Interpretation. *Econometrica 50*(1), 213–224.

Hellwig, C., S. Kohls, and L. Veldkamp (2012, May). Information choice technologies. *American Economic Review 102*(3), 35–40.

Hoberg, G. and V. Maksimovic (2019). Product Life Cycles in Corporate Finance. *Working Paper*.

Jazwinski, A. (1970). *Stochastic Processes and Filtering Theory*. Academic Press, New York.

Jovanovic, B. (1982). Selection and the evolution of industry. *Econometrica 50*(3), 649–670.

Jovanovic, B. and Y. Nyarko (1996, November). Learning by Doing and the Choice of Technology. *Econometrica 64*(6), 1299–1310.

Jovanovic, B. and R. Rob (1990). Long waves and short waves: Growth through intensive and extensive search. *Econometrica 58*(6), 1391–1409.

Judd, K. L. (1998, January). *Numerical Methods in Economics*, Volume 1 of *MIT Press Books*. The MIT Press.

Keller, G. and S. Rady (1999). Optimal Experimentation in a Changing Environment. *Review of Economic Studies 66*(3), 475–507.

Kyle, A. S., A. A. Obizhaeva, and Y. Wang (2018). Smooth Trading with Overconfidence and Market Power. *Review of Economic Studies 85*(1), 611–662.

Lipster, R. S. and A. N. Shiryaev (2001). *Statistics of Random Processes II*. Springer Verlag, New York.

Manso, G. (2011). Motivating innovation. *The Journal of Finance 66*(5), 1823–1860.

March, J. G. (1991). Exploration and exploitation in organizational learning. *Organization Science 2*(1), 71–87.

Moscarini, G. and L. Smith (2001). The optimal level of experimentation. *Econometrica 69*(6), 1629–1644.

Pastor, L. and P. Veronesi (2009). Technological revolutions and stock prices. *The American Economic Review 99*(4), 1451–1483.

Peters, R. H. and L. A. Taylor (2017). Intangible capital and the investment-q relation. *Journal of Financial Economics 100*, 251–272.

Prescott, E. C. and M. Visscher (1980). Organization capital. *Journal of Political Economy* (88), 446–461.

Rob, R. (1991). Learning and capacity expansion under demand uncertainty. *The Review of Economic Studies 58*(4), 655–675.

Stokey, N. and R. Lucas (1996). *Recursive methods in economic dynamics* (4. print ed.). Cambridge, Mass. [u.a.]: Harvard Univ. Press.

Thomke, S. H. (1998). Managing experimentation in the design of new products. *Management Science 44*(6), 743–762.

van Nieuwerburgh, S. and L. Veldkamp (2010). Information acquisition and under-diversification. *The Review of Economic Studies 77*(2), 779–805.

Wald, A. (1947). Foundations of a general theory of sequential decision functions. *Econometrica 15*(4), 279–313.

Weitzman, M. (1979). Optimal search for the best alternative. *Econometrica 47*(3), 641–54.

A General solution in the inaction region

In the inaction region, the processes for knowledge and physical capital move of their own accord. As a result, the firm value in Eq. (6) satisfies the standard HJB equation, which we omit for the sake of brevity. Substituting the functional form in Eq. (7) into this equation, we obtain an ODE for the intensive value, $v(\cdot)$, that it must satisfy within the inaction region:

$$\left(r + \eta Z^2 \phi - (\eta - 1)\left(\alpha\delta + \frac{\eta}{2} - \sqrt{\tau_A}Z\right)\right)v = 1 + \sqrt{\tau_A}v'\left(\frac{\sqrt{\tau_A}}{2}Z + 1 - \eta\right) + \frac{\tau_A}{2}v'', \ Z \in \mathscr{I} \quad (32)$$

which is a second-order inhomogeneous linear equation.

Clearly, the firm could always choose never to invest nor to innovate, and this outcome must thus be a particular solution, which we denote v_P, of this ODE. In particular, the value of the firm when it remains inactive forever is

$$V(N_t, A_t, K_t, \widehat{M_t}, \Omega_t) \equiv \mathbb{E}\left[\int_0^\infty e^{-rs}\Pi(A_{t+s}, K_{t+s}, N_{t+s})\mathrm{d}s \,\Big|\, \mathscr{F}_t\right]$$

$$= \Pi(A_t, K_t, N_t)\mathbb{E}\left[\int_0^\infty e^{\int_t^{t+s}\left((1-\eta)\left(\tau_A^{1/2}Z_u - \alpha\delta - 1/2\right) - \eta\phi Z_u^2 - r\right)\mathrm{d}u + (1-\eta)(\widehat{B}_{t+s} - \widehat{B}_t)}\mathrm{d}s \,\Big|\, \mathscr{F}_t\right].$$

Introducing the change of probability measure,

$$\left.\frac{\widetilde{\mathbb{P}}}{\widehat{\mathbb{P}}}\right|_{\mathscr{F}_t} = e^{-1/2(1-\eta)^2 t + (1-\eta)B_t},$$

and applying Fubini's theorem, we write this value as:

$$V(N_t, A_t, K_t, \widehat{M_t}, \Omega_t) = \Pi(A_t, K_t, N_t)\int_0^\infty \widetilde{\mathbb{E}}\left[e^{\int_t^{t+s}\left((1-\eta)\left(\tau_A^{1/2}Z_u - \alpha\delta - \eta/2\right) - \eta\phi Z_u^2 - r\right)\mathrm{d}u} \,\Big|\, \mathscr{F}_t\right]\mathrm{d}s$$

$$\equiv \Pi(A_t, K_t, N_t)v_P(Z_t);$$

accordingly, we define the function

$$f(s, Z) = \widetilde{\mathbb{E}}\left[e^{\int_t^{t+s}\left((1-\eta)\left(\tau_A^{1/2}Z_u - \alpha\delta - \eta/2\right) - \eta\phi Z_u^2 - r\right)du}\,\middle|\,\mathscr{F}_t\right],$$

which, expressing the dynamics of Z under $\widetilde{\mathbb{P}}$, satisfies the following PDE:

$$f_s = \left((1-\eta)\left(\tau_A^{1/2}Z_u - \alpha\delta - \eta/2\right) - \eta\phi Z_u^2 - r\right)f + (\tau_A^{1/2}(1-\eta) + \tau_A/2Z)f_Z + \tau_A/2 f_{ZZ},$$

with boundary condition, $f(0, Z) \equiv 1$. Since Z is a Gaussian process, the solution takes the usual exponential affine quadratic form:

$$f(s, Z) = \exp\left(a_0(s) + a_1(s)Z + a_2(s)Z^2\right),$$

with coefficients satisfying the system of Riccati equations:

$$a_0' = (\eta - 1)\left(\alpha\delta + \frac{\eta}{2}\right) + \frac{a_1^2\tau_A}{2} + a_1(1-\eta)\sqrt{\tau_A} + a_2\tau_A - r,$$
$$a_1' = a_1(4a_2\tau_A + \tau_A) + (2a_2 + 1)(1-\eta)\sqrt{\tau_A}$$
$$a_2' = -\eta\phi + a_2(2a_2 + 1)\tau_A,$$

with boundary conditions, $a_0(0) = a_1(0) = a_2(0) = 0$. These equations have closed-form solutions of the form:

$$a_2(s) = \frac{1}{4}\left(-\frac{\xi\tanh\left(\frac{1}{2}\xi s\sqrt{\tau_A} - \coth^{-1}\left(\frac{\xi}{\sqrt{\tau_A}}\right)\right)}{\sqrt{\tau_A}} - 1\right), \tag{33}$$

$$a_1(s) = \frac{(\eta - 1)\left(\left(\xi + \sqrt{\tau_A}\right)^2 - 2\left(\xi^2 + \tau_A\right)e^{\frac{1}{2}\xi s\sqrt{\tau_A}} + \left(\xi - \sqrt{\tau_A}\right)^2 e^{\xi s\sqrt{\tau_A}}\right)}{\xi\sqrt{\tau_A}\left(\xi + \left(\xi - \sqrt{\tau_A}\right)e^{\xi s\sqrt{\tau_A}} + \sqrt{\tau_A}\right)},$$

$$a_0(s) = -\frac{4\eta^2 + s\tau_A(4\alpha\delta + 4r + \tau_A + 2) - 2\eta(s(2\alpha\delta\tau_A + \tau_A) + 4) - \tau_A\log(4) + 4}{4\tau_A}$$

$$+ \frac{(\eta-1)^2(s\tau_A + 2)}{2\xi^2\tau_A} + \frac{1}{4}\xi s\sqrt{\tau_A} + \frac{1}{2}\left(\log(\xi) - \log\left(\xi + \left(\xi - \sqrt{\tau_A}\right)e^{\xi s\sqrt{\tau_A}} + \sqrt{\tau_A}\right)\right)$$

$$\frac{(\eta - 1)^2\left(\left(\xi^4 + \xi^2\tau_A + 2\tau_A^2\right)\cosh\left(\frac{1}{2}\xi s\sqrt{\tau_A}\right) - 2\tau_A\left(\xi^2 + \tau_A\right)\right)}{\xi^3\tau_A\left(\xi\cosh\left(\frac{1}{2}\xi s\sqrt{\tau_A}\right) - \sqrt{\tau_A}\sinh\left(\frac{1}{2}\xi s\sqrt{\tau_A}\right)\right)},$$

where $\xi \equiv \sqrt{\tau_A + 8\eta\phi}$. We conclude that a particular solution to Eq. (32) is as in Eq. (15). Furthermore, the two solutions associated with the homogeneous version of Eq. (32) are:

$$G_1(Z) = e^{g_1 Z + g_2 Z^2}H_n\left(h_0 + h_1 Z\right),$$

with $g_1 = (\eta - 1)\left(\frac{1}{\sqrt{\tau_A}} - \frac{1}{\xi}\right)$, $g_2 = -\frac{\sqrt{\tau_A} + \xi}{4\sqrt{\tau_A}}$, $h_0 = \frac{\sqrt{2}(\eta - 1)\sqrt[4]{\tau_A}}{\xi^{3/2}}$, $h_1 = \sqrt{\frac{\xi}{2\sqrt{\tau_A}}}$, and H_n denotes the

Hermite polynomial of order n (as defined in Eq. (16)) and

$$G_2(Z) = e^{g_1 Z + g_2 Z^2} M\left(-\frac{1}{2}n, \frac{1}{2}, m_0 + m_1 Z + m_2 Z^2\right),$$

with $m_0 = \frac{2(\eta-1)^2 \sqrt{\tau_A}}{\xi^3}$, $m_1 = \frac{2(\eta-1)}{\xi}$ and $m_2 = \frac{\xi}{2\sqrt{\tau_A}}$, where M denotes the Kummer confluent hypergeometric function. We conclude that, in the inaction region, the general solution to Eq. (32) takes the form in Eq. (14) Note that the existence of this solution depends on several conditions.

B Expected firm value upon experimentation

It is convenient to first define the firm value outside $\mathscr{B}$ as:

$$h(Z) \equiv g(Z; \underline{C}_1, \underline{C}_2)\mathbf{1}_{Z \in (\underline{a}, \underline{b})} + (1-\omega)^{\alpha(1-\eta)} V(0)\mathbf{1}_{Z \notin \mathscr{A}} + g(Z; \overline{C}_1, \overline{C}_2)\mathbf{1}_{Z \in (\overline{b}, \overline{a})}.$$

Letting the (yet unknown) optimal investment policy $i(Z)$ be a function of the current state Z, we can rewrite the firm value as:

$$v(Z) \equiv h(Z) + \mathbf{1}_{Z \in \mathscr{B}}\left((1+i(Z))^{\alpha(1-\eta)}\int_{\mathbb{R}} v(y)\phi(y; Z, i(Z))\mathrm{d}y - (\kappa + \gamma/2i(Z)^2)\right),$$

To alleviate notation, we further let for each experimentation round k:

$$i_k \equiv i(y^{(k)}), \quad k \in \mathbb{N},$$

and define:

$$\phi^{(n)}(y; Z, \{i_k\}_{k=0}^n) := \int_{\mathscr{B}^n} \prod_{k=1}^n (1+i_k)^{\alpha(1-\eta)}\phi(y; y^{(n)}, i_n)\mathrm{d}\Phi(y^{(n)}; y^{(n-1)}, i_{n-1})\cdots\mathrm{d}\Phi(y^{(1)}; y^{(0)} = Z, i_0),$$

(34)

with $\phi^{(0)}(y; Z) \equiv \phi(y; Z)$. With this notation at hand and assuming for the moment that there is a contraction in the sense that

$$\lim_{n\to\infty} \phi^{(n)}(y; Z, \{i_k\}_{k=0}^n) = 0, \quad \forall y, Z,$$

we can then express the expected firm value as

$$\int_{\mathbb{R}} v(y)\phi(y; Z, i(Z))\mathrm{d}y = \sum_{n=0}^{\infty} \int_{\mathbb{R}} \left(h(y) - (\kappa + \gamma/2i(y)^2)\mathbf{1}_{y\in\mathscr{B}}\right)\phi^{(n)}(y; Z, \{i_k\}_{k=0}^n)\mathrm{d}y.$$

It follows that for $Z \in \mathscr{B}$ the firm faces the following dynamic programming problem:

$$V(Z) := \max_{\{i_k\}_{k=0}^{\infty}} (1+i_0)^{\alpha(1-\eta)} \sum_{n=0}^{\infty} \int_{\mathbb{R}} \left(h(y) - (\kappa + \gamma/2i(y)^2) \mathbf{1}_{y \in \mathscr{B}} \right) \phi^{(n)}(y; Z, \{i_k\}_{k=0}^n) \mathrm{d}y - (\kappa + \gamma/2i_0^2).$$

$$(35)$$

As is customary we wish to formulate this program recursively in the form of a Bellman equation:

$$V(Z) \equiv \max_{i_0} (1+i_0)^{\alpha(1-\eta)} \int_{\mathbb{R}} h(y)\phi(y; Z, i_0)\mathrm{d}y - (\kappa + \gamma/2i_0^2)$$

$$+ \max_{\{i_k\}_{k=0}^{\infty}} (1+i_0)^{\alpha(1-\eta)} \left(\sum_{n=1}^{\infty} \int_{\mathbb{R}} h(y)\phi^{(n)}(y; Z, \{i_k\}_{k=0}^n)\mathrm{d}y - \sum_{n=0}^{\infty} \int_{\mathscr{B}} (\kappa + \gamma/2i(y)^2)\phi^{(n)}(y; Z, \{i_k\}_{k=0}^n)\mathrm{d}y \right).$$

Rewriting Eq. (34) recursively as:

$$\phi^{(n+1)}(y; Z, \{i_k\}_{k=0}^{n+1}) = \int_{\mathscr{B}} (1+i(x))^{\alpha(1-\eta)} \phi^{(n)}(y; Z, \{i_k\}_{k=1}^n)\phi(x; Z, i_0)\mathrm{d}x,$$

we can rewrite the first term in brackets as:

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}} h(y)\phi^{(n)}(y; Z, \{i_k\}_{k=0}^n)\mathrm{d}y = \sum_{n=0}^{\infty} \int_{\mathbb{R}} h(y)\phi^{(n+1)}(y; Z, \{i_k\}_{k=0}^{n+1})\mathrm{d}y$$

$$= \int_{\mathscr{B}} (1+i(x))^{\alpha(1-\eta)} \sum_{n=0}^{\infty} \int_{\mathbb{R}} h(y)\phi^{(n)}(y; Z, \{i_k\}_{k=0}^n)\mathrm{d}y \phi(x; Z, i_0)\mathrm{d}x.$$

and the second term in brackets as:

$$\sum_{n=0}^{\infty} \int_{\mathscr{B}} (\kappa + \gamma/2i(y)^2)\phi^{(n)}(y; Z, \{i_k\}_{k=0}^n)\mathrm{d}y$$

$$= \int_{\mathscr{B}} \left(\kappa + \gamma/2i(x)^2 + (1+i(x))^{\alpha(1-\eta)} \sum_{n=0}^{\infty} \int_{\mathscr{B}} (\kappa + \gamma/2i(y)^2)\phi^{(n)}(y; x, \{i_k\}_{k=0}^n)\mathrm{d}y \right) \phi(x; Z, i_0)\mathrm{d}x.$$

47

We then substitute this expression in the value function to obtain:

$$V(Z) \equiv \max_{i_0}(1+i_0)^{\alpha(1-\eta)}\int_{\mathbb{R}} h(y)\phi(y;Z,i_0)\mathrm{d}y - \kappa - \gamma/2i_0^2 + \max_{\{i_k\}_{k=0}^{\infty}}(1+i_0)^{\alpha(1-\eta)}\times \tag{36}$$

$$\int_{\mathscr{B}}\left(\begin{array}{c}(1+i(x))^{\alpha(1-\eta)}\sum_{n=0}^{\infty}\int_{\mathbb{R}}\left(h(y)-(\kappa+\gamma/2i(y)^2)\mathbf{1}_{y\in\mathscr{B}}\right)\phi^{(n)}(y;x,\{i_k\}_{k=0}^{n})\mathrm{d}y\\-\kappa-\gamma/2i(x)^2\end{array}\right)\phi(x;Z,i_0)\mathrm{d}x$$

$$= \max_{i}(1+i)^{\alpha(1-\eta)}\int_{\mathbb{R}}h(y)\phi(y;Z,i)\mathrm{d}y - \kappa - \gamma/2i^2 + (1+i)^{\alpha(1-\eta)}\int_{\mathscr{B}}\phi(x;Z,i)V(x)\mathrm{d}x,$$

where the last equation yields the Bellman equation we are looking for. We are further looking for an optimal policy function $i(\cdot)$ such that

$$V(z) = (1+i(Z))^{\alpha(1-\eta)}\int_{\mathbb{R}}h(y)\phi(y;Z,i)\mathrm{d}y - \kappa - \gamma/2i(Z)^2 + (1+i(Z))^{\alpha(1-\eta)}\int_{\mathscr{B}}\phi(x;Z,i)V(x)\mathrm{d}x.$$

Differentiating Eq. (36), the optimal policy function must satisfy the first-order condition:

$$0 = \frac{\alpha(1-\eta)}{1+i(Z)}V(Z) + \frac{\alpha(1-\eta)(\kappa+\gamma/2i(Z)^2)-\gamma(1+i(Z))i(Z)}{1+i(Z)} \tag{37}$$

$$+ (1+i(Z))^{\alpha(1-\eta)}\left(\int_{\mathbb{R}}h(y)\frac{\partial}{\partial i}\phi(x;i,Z)\mathrm{d}y + \int_{\mathscr{B}}V(y)\frac{\partial}{\partial i}\phi(x;i,Z)\mathrm{d}y\right).$$

We now want to prove that a solution to the Bellman equation exists. We start by computing the first expectation in Eq. (36). It is convenient to define

$$\psi(Z;i,c_0,c_1,c_2,\underline{x},\overline{x}) := \int_{\underline{x}}^{\overline{x}}\exp\left(c_0+c_1y+c_2y^2\right)\phi(y;i(Z),Z)\mathrm{d}y$$

$$= \frac{e^{c_0+\frac{c_1^2 i(Z)\tau_S+2Z(c_1\sqrt{1+\tau_S i}+c_2 Z(1+\tau_S i(Z)))}{2i\tau_S(1-2c_2\tau_S i(Z))}}}{\sqrt{2(1-2c_2\tau_S i(Z))}}\left(\begin{array}{c}\operatorname{erf}\left(\frac{c_1\tau_S i(Z)+Z\sqrt{1+\tau_S i(Z)}-\overline{x}(1-2c_2\tau_s i(Z))}{\sqrt{2\tau_S i(Z)(1-2c_2\tau_S i(Z))}}\right)\\-\operatorname{erf}\left(\frac{c_1\tau_S i(Z)+Z\sqrt{1+\tau_S i(Z)}-\underline{x}(1-2c_2\tau_s i(Z))}{\sqrt{2\tau_S i(Z)(1-2c_2\tau_S i(Z))}}\right)\end{array}\right),$$

where c_0, c_1, c_2, and $\overline{x} \geq \underline{x}$ are constant coefficients. We note that this function exists provided that

$$c_2 < \frac{1}{2\tau_S i(Z)}, \tag{38}$$

which imposea an upper bound on the investment policy function $i(Z)$ when $c_2 > 0$. Using this

function and applying Fubini's theorem it follows that

$$\int_{\underline{x}}^{\overline{x}} v_P(y)\,\phi(y; i(Z), Z)\mathrm{d}y = \int_0^\infty \psi(Z; i(Z), a_0(s), a_1(s), a_2(s), \underline{x}, \overline{x})\mathrm{d}s =: \overline{v}_P(Z, \underline{x}, \overline{x}).$$

To compute the expectation of the second term involved in Eq. (14), we use the integral representation of the Hermite polynomial. Note that the order of the polynomial in Eq. (16) may be negative, in which case the following representation is appropriate:

$$H_n(x) \equiv \frac{(-2\mathring{\imath})^{-n}}{\sqrt{\pi}}\int_{\mathbb{R}} t^{-n}\exp(-(t - \mathring{\imath}x)^2)\mathrm{d}t.$$

We then obtain:

$$\int_{\underline{x}}^{\overline{x}} G_1(y)\,\phi(y; i(Z), Z)\mathrm{d}y = \int_{\underline{x}}^{\overline{x}} e^{g_1 y + g_2 y^2} H_n(h_0 + h_1 y)\,\phi(y; i(Z), Z)\mathrm{d}y$$

$$= \int_{\underline{x}}^{\overline{x}} e^{g_1 y + g_2 y^2}\frac{(-2\mathring{\imath})^{-n}}{\sqrt{\pi}}\int_{\mathbb{R}} t^{-n}\exp\left(-(t - \mathring{\imath}(h_0 + h_1 y))^2\right)\mathrm{d}t\,\phi(y; i(Z), Z)\mathrm{d}y$$

$$= \frac{(-2\mathring{\imath})^{-n}}{\sqrt{\pi}}\int_{\mathbb{R}} t^{-n}\psi(Z; i(Z), (h_0 + \mathring{\imath}t)^2, g_1 + 2h_1(h_0 + \mathring{\imath}t), g_2 + h_1^2, \underline{x}, \overline{x})\mathrm{d}t$$

$$=: \overline{G}_1(Z, \underline{x}, \overline{x}),$$

where the last equality uses Fubini's theorem to slide the expectation inside the integral. Similarly, using the integral representation of the confluent hypergeometric function we further obtain:

$$\int_{\underline{x}}^{\overline{x}} G_2(y)\,\phi(y; i(Z), Z)\mathrm{d}y = \int_{\underline{x}}^{\overline{x}} e^{g_1 y + g_2 y^2} M\left(\frac{1}{2}n, \frac{1}{2}, m_0 + m_1 y + m_2 y^2\right)\mathrm{d}y$$

$$= \int_{\underline{x}}^{\overline{x}} e^{g_1 y + g_2 y^2}\int_0^1 \frac{t^{n/2-1}(1-t)^{-\frac{n+1}{2}}}{\Gamma(n/2)\Gamma((1-n)/2)}e^{(m_0 + m_1 y + m_2 y^2)t}\mathrm{d}t\mathrm{d}y$$

$$= \int_0^1 \frac{t^{n/2-1}(1-t)^{-\frac{n+1}{2}}}{\Gamma(n/2)\Gamma((1-n)/2)}\psi(Z; i(Z), m_0 t, g_1 + m_1 t, g_2 + m_2 t, \underline{x}, \overline{x})\mathrm{d}t$$

$$=: \overline{G}_2(Z, \underline{x}, \overline{x}),$$

where the last equality uses Fubini's theorem. Using these expressions we can write the expectation

in Eq. (36) as:

$$\int_{\mathbb{R}} h(y)\phi(y; Z, i)\mathrm{d}y = \overline{v}_P(Z, \underline{a}, \underline{b}) + \underline{C}_1\overline{G}_1(Z, \underline{a}, \underline{b}) + \underline{C}_2\overline{G}_2(Z, \underline{a}, \underline{b}) + \overline{v}_P(Z, \overline{b}, \overline{a}) + \overline{C}_1\overline{G}_1(Z, \overline{b}, \overline{a})$$

$$+ \overline{C}_2\overline{G}_2(Z, \overline{b}, \overline{a}) + (1 - \delta)^{\alpha(1-\eta)}V(0)\int_{y\notin\mathscr{A}} \phi(y; i(Z), Z)\mathrm{d}y.$$

Using the existence condition in Eq. (38), the existence of this expectation depends on the following conditions being satisfied:

$$a_2(s)\tau_S i(Z) < 1/2, \quad \forall s > 0$$

$$\tau_S i(Z)\left(-\frac{1}{2} + \left(y - \frac{1}{2}\right)\frac{\xi}{\sqrt{\tau_A}}\right) < 1, \quad \forall y \in (0, 1)$$

These conditions place an upper bound on the investment rate. In particular, the last condition is always satisfied provided that the investment rate satisfies:

$$i < \frac{2}{\tau_S(\xi/\sqrt{\tau_A} - 1)} =: \overline{i}. \tag{39}$$

Furthermore, using Eq. (33) for a_2, since $\tanh : \mathbb{R} \to [-1, 1]$ it follows that the first equation is always satisfied if the condition in Eq. (39) is satisfied; thus it is a sufficient condition for the expectations above to be well-defined. We will thus take the admissible investment set to be

$$\mathscr{D} := [0, \overline{i}).$$

C Solution method

We start by imposing that the firm value be continuous over each region involved in its piecewise representation in Eq. (21). In particular, when the firm invests there can no jump in its value on average;

$$g(\underline{b}; \underline{C}_1, \underline{C}_2) = V(\underline{b}), \tag{40}$$
$$g(\overline{b}; \overline{C}_1, \overline{C}_2) = V(\overline{b}).$$

Similarly, there can be no jump in the firm value when it decides to explore:

$$g(\underline{a}; \underline{C}_1, \underline{C}_2) = V(0)(1 - \delta)^{\alpha(1-\eta)}, \tag{41}$$
$$g(\overline{a}; \overline{C}_1, \overline{C}_2) = V(0)(1 - \delta)^{\alpha(1-\eta)}.$$

Eqs. (40)–(41) constitute a system of four equations that determines the four unknown constant of integration, $\underline{C}_1$, $\underline{C}_2$, $\overline{C}_1$, and $\overline{C}_2$.

A difficulty in solving this system of equations is that the functional form of the value function, $V(\cdot)$, is unknown. There are two alternative routes in handling this problem numerically. One is to use the representation of the value function as infinite sum in Eq. (35). Another, more appropriate way is to use Gauss-Legendre quadrature (see, e.g., Judd (1998)) to obtain an explicit representation of the value function. This latter approach is the one we adopt. In particular, we approximate the integral involved in the Bellman equation (22) as

$$\int_{\mathscr{B}} V(y)\phi(y; i(z), z)\mathrm{d}y \approx \sum_{k=1}^{n} \omega_k \phi(x_k; i(z), z)V(x_k),$$

where x_k are the quadrature nodes and ω_k are the quadrature weights over the interval $\mathscr{B}$. We then collect the value function evaluated at each quadrature node in a vector, which we denote by $\mathbf{V} \equiv \{V(x_j)\}_{j=1}^{n}$. The goal is now to solve for this vector explicitly. To do so, we plug the integral approximation back into the Bellman equation and evaluate it at the quadrature nodes:

$$V(z_j) = (1 + i(z_j))^{\alpha(1-\eta)} \int_{\mathbb{R}} (h(y) - \mathbf{1}_{y\in\mathscr{B}}(\kappa + \gamma/2i(y)^2))\phi(y; i(z_j), z_j)\mathrm{d}y \qquad (42)$$

$$- \kappa - \gamma/2i(z_j)^2 + (1 + i(z_j))^{\alpha(1-\eta)} \sum_{k=1}^{n} \omega_k \phi(x_k; i(z_j), z_j)V(x_k), \quad j = 1, \ldots, n,$$

which yields a system of n equations for the n unknown elements composing the vector $\mathbf{V}$.

We then define the following (column) vectors:

$$\boldsymbol{\Phi} \equiv \left((1 + i(z_j))^{\alpha(1-\eta)}(\overline{v}_P(z_j, \underline{a}, \underline{b}) + \overline{v}_P(z_j, \overline{b}, \overline{a})) - \kappa - \gamma/2i(z_j)^2\right)_{j=1,\ldots,n},$$

$$\underline{\mathbf{G}}_k \equiv \left((1 + i(z_j))^{\alpha(1-\eta)}(\overline{G}_k(z_j, \underline{a}, \underline{b}))\right)_{j=1,\ldots,n}, \quad k = 1, 2,$$

$$\overline{\mathbf{G}}_k \equiv \left((1 + i(z_j))^{\alpha(1-\eta)}(\overline{G}_k(z_j, \overline{b}, \overline{a}))\right)_{j=1,\ldots,n}, \quad k = 1, 2,$$

$$\mathbf{A} \equiv \left((1 - \omega)^{\alpha(1-\eta)} \int_{y\notin(\underline{a},\overline{a})} \phi(y; i(z_j), z_j)\mathrm{d}y\right)_{j=1,\ldots,n},$$

and the following $n \times n$ matrix:

$$\boldsymbol{\Sigma} \equiv \left((1 + i(z_j))^{\alpha(1-\eta)}\omega_i\phi(x_i; i(z_j), z_j)\right)_{i,j=1,\ldots,n}.$$

We can then rewrite the system in Eq. (42) in vector form and express the vector $\mathbf{V}$ as:

$$\mathbf{V} = (\mathbf{I} - \boldsymbol{\Sigma})^{-1} \left(\boldsymbol{\Phi} + \sum_{k=1,2} \underline{C}_k \underline{\mathbf{G}}_k + \sum_{k=1,2} \overline{C}_k \overline{\mathbf{G}}_k + \mathbf{A}V(0)\right).$$

By analogy to the vectors we defined above, we further define the following functions:

$$\Phi(Z) \equiv (1+i(Z))^{\alpha(1-\eta)}(\overline{v}_P(Z,\underline{a},\underline{b}) + \overline{v}_P(Z,\overline{b},\overline{a})) - \kappa - \gamma/2i(Z)^2,$$

$$\underline{G}_k(Z) \equiv (1+i(Z))^{\alpha(1-\eta)}(\overline{G}_k(Z,\underline{a},\underline{b}), \quad k=1,2,$$

$$\overline{G}_k(Z) \equiv (1+i(Z))^{\alpha(1-\eta)}(\overline{G}_k(Z,\overline{b},\overline{a}), \quad k=1,2,$$

$$A(Z) \equiv (1-\omega)^{\alpha(1-\eta)} \int_{y \notin (\underline{a},\overline{a})} \phi(y;i(Z),Z)\mathrm{d}y,$$

and the following vector:

$$\Sigma(Z) \equiv \left((1+i(Z))^{\alpha(1-\eta)} \omega_i \phi(x_i;i(Z),Z) \right)_{i=1,\dots,n}.$$

We use these functions to write the Nystrom extension (see, e.g., Judd (1998)) to the value function as:

$$V(Z) \approx \Phi(Z) + \sum_{k=1,2} \underline{C}_k \underline{G}_k(Z) + \sum_{k=1,2} \overline{C}_k \overline{G}_k(Z) + A(Z)V(0) +$$

$$\Sigma(Z)' \left(\mathbf{I} - \boldsymbol{\Sigma}\right)^{-1} \left(\boldsymbol{\Phi} + \sum_{k=1,2} \underline{C}_k \underline{\mathbf{G}}_k + \sum_{k=1,2} \overline{C}_k \overline{\mathbf{G}}_k + \mathbf{A}V(0) \right).$$

Evaluating the Nystrom extension at 0 yields:

$$V(0) = \frac{1}{1 - A(0) - \Sigma(0)'(\mathbf{I}-\boldsymbol{\Sigma})^{-1}\mathbf{A}} \left(\begin{array}{c} \Phi(0) + \sum_{k=1,2}\underline{C}_k\underline{G}_k(0) + \sum_{k=1,2}\overline{C}_k\overline{G}_k(0) \\ +\Sigma(0)'(\mathbf{I}-\boldsymbol{\Sigma})^{-1}\left(\boldsymbol{\Phi} + \sum_{k=1,2}\underline{C}_k\underline{\mathbf{G}}_k + \sum_{k=1,2}\overline{C}_k\overline{\mathbf{G}}_k\right) \end{array} \right).$$

Plugging back in the Nystrom extension and defining:

$$\phi_0(Z) \equiv \Phi(Z) + \Sigma(Z)'(\mathbf{I}-\boldsymbol{\Sigma})^{-1}\boldsymbol{\Phi} + \frac{A(Z) + \Sigma(Z)'(\mathbf{I}-\boldsymbol{\Sigma})^{-1}\mathbf{A}}{1 - A(0) - \Sigma(0)'(\mathbf{I}-\boldsymbol{\Sigma})^{-1}\mathbf{A}}\left(\Phi(0) + \Sigma(0)'(\mathbf{I}-\boldsymbol{\Sigma})^{-1}\boldsymbol{\Phi}\right),$$

$$\underline{g}_k(Z) \equiv \underline{G}_k(Z) + \Sigma(Z)'(\mathbf{I}-\boldsymbol{\Sigma})^{-1}\underline{\mathbf{G}}_k + \frac{A(Z) + \Sigma(Z)'(\mathbf{I}-\boldsymbol{\Sigma})^{-1}\mathbf{A}}{1 - A(0) - \Sigma(0)'(\mathbf{I}-\boldsymbol{\Sigma})^{-1}\mathbf{A}}\left(\underline{G}_k(0) + \Sigma(0)'(\mathbf{I}-\boldsymbol{\Sigma})^{-1}\underline{\mathbf{G}}_k\right),$$

$$\overline{g}_k(Z) \equiv \overline{G}_k(Z) + \Sigma(Z)'(\mathbf{I}-\boldsymbol{\Sigma})^{-1}\overline{\mathbf{G}}_k + \frac{A(Z) + \Sigma(Z)'(\mathbf{I}-\boldsymbol{\Sigma})^{-1}\mathbf{A}}{1 - A(0) - \Sigma(0)'(\mathbf{I}-\boldsymbol{\Sigma})^{-1}\mathbf{A}}\left(\overline{G}_k(0) + \Sigma(0)'(\mathbf{I}-\boldsymbol{\Sigma})^{-1}\overline{\mathbf{G}}_k\right),$$

, provides an approximate but explicit representation of the value function in terms of the four

unknown integration constants:

$$V(Z) \approx \phi_0(Z) + \sum_{k=1,2} \underline{C}_k \underline{g}_k(Z) + \sum_{k=1,2} \overline{C}_k \overline{g}_k(Z).$$

Using this representation, and letting $\widehat{\omega} \equiv (1-\omega)^{\alpha(1-\eta)}$ we define the following matrix:

$$B = \begin{pmatrix} G_1(\underline{a}) - \widehat{\omega}\underline{g}_1(0) & G_2(\underline{a}) - \widehat{\omega}\underline{g}_2(0) & -\widehat{\omega}\overline{g}_2(0) & -\widehat{\omega}\overline{g}_2(0) \\ -\widehat{\omega}\underline{g}_1(0) & -\widehat{\omega}\underline{g}_2(0) & G_1(\overline{a}) - \widehat{\omega}\overline{g}_2(0) & G_2(\overline{a}) - \widehat{\omega}\overline{g}_2(0) \\ G_1(\underline{b}) - \underline{g}_1(\underline{b}) & G_2(\underline{b}) - \underline{g}_2(\underline{b}) & -\overline{g}_2(\underline{b}) & -\overline{g}_2(\underline{b}) \\ -\underline{g}_1(\overline{b}) & -\underline{g}_2(\overline{b}) & G_1(\overline{b}) - \overline{g}_2(\overline{b}) & G_2(\overline{b}) - \overline{g}_2(\overline{b}) \end{pmatrix}.$$

we can in turn write the solution of the system in Eqs. (40)–(41) explicitly as:

$$\begin{pmatrix} \underline{C}_1 \\ \underline{C}_2 \\ \overline{C}_1 \\ \overline{C}_2 \end{pmatrix} = B^{-1} \begin{pmatrix} \phi_0(0)\widehat{\omega} - v_P(\underline{a}) \\ \phi_0(0)\widehat{\omega} - v_P(\overline{a}) \\ \phi_0(\underline{b}) - v_P(\underline{b}) \\ \phi_0(\overline{b}) - v_P(\overline{b}) \end{pmatrix}.$$

This solution is unique if and only if the determinant of $\det(B) \neq 0$.

The solution we have constructed holds for given thresholds $\underline{a}$, $\overline{a}$, $\underline{b}$, and $\overline{b}$, and a given knolwedge-contingent investment policy $i(Z)$. The last step is to pick these elements optimally. To solve for the optimal thresholds we use the system of smooth-pasting condition in Eqs. (13)–(23), wich provides four equations for the four unknowns, $\underline{a}$, $\overline{a}$, $\underline{b}$, and $\overline{b}$. The optimal investment policy further solves the first-order condition in Eq. (37). To find a solution, we use a Newton algorithm to solve for Eqs. (13)–(23), and at each Newton step we perform Policy Function Iteration to obtain the optimal investment policy.

D Stationary Distribution of Knowledge

We derive the stationary distribution of knowledge based on its Kolmogorov Backward Equation (KBE). Define the cumulative probability that Z_T is below level y at time T given that it started at level x at time t as:

$$\phi(y, x, T-t) := \mathbb{E}\left[\mathbf{1}_{Z_T \leq y} | Z_t = x\right].$$

This expression is a martingale, which implies that in the inaction regions this function satisfies the PDE:

$$\phi_t = \tfrac{\tau_A}{2}(\phi_x x + \phi_{xx}), \quad \phi(y, x, 0) = \mathbf{1}_{x \leq y}, \quad Z \in (\underline{a}, \underline{b}) \cup (\overline{b}, \overline{a}) . \tag{43}$$

Furthermore, for this expression to be a martingale at $Z = \underline{a}$, $Z \equiv \overline{a}$, and over the experimentation region $Z \in \mathscr{B}$, we must further ensure that there is no jump in the function at these points and

over the experimentation region:

$$\phi(y, \bar{a}, T - t) = \phi(y, \underline{a}, T - t) = \phi(y, 0, T - t) \tag{44}$$

and

$$\phi(y, Z, T - t) = \int_{\underline{a}}^{\bar{a}} \phi(y, x, T - t)\varphi(x, i(Z), Z)\mathrm{d}x + \phi(y, 0, T - t) \int_{x \notin (\underline{a}, \bar{a})} \varphi(x, i(Z), Z)\mathrm{d}x, \quad Z \in \mathscr{B}, \tag{45}$$

where $\varphi(\cdot)$ denotes the normal density with mean $\sqrt{1 + \tau_S i(Z)}Z$ and variance $\tau_S i(Z)$.

Since we are interested in deriving the stationary distribution, we introduce the following time transform:

$$\psi(y, x; \lambda) \equiv \int_0^{+\infty} \exp(-\lambda\tau)\phi(y, x, \tau)\mathrm{d}\tau.$$

The Kolmogorov Backward Equation (43) becomes an ODE:

$$0 = \lambda(\mathbf{1}_{x \leq y} - \psi) + \frac{\tau_A}{2}(\psi_x x + \psi_{xx}), \quad x \in (\underline{a}, \underline{b}) \cup (\bar{b}, \bar{a}) \tag{46}$$

and Conditions (44)–(45) become:

$$\psi(y, \bar{a}; \lambda) = \psi(y, \underline{a}; \lambda) = \psi(y, 0; \lambda) \tag{47}$$

and

$$\psi(y, Z; \lambda) = \int_{\underline{a}}^{\bar{a}} \psi(y, x; \lambda)\varphi(x, i(Z), Z)\mathrm{d}x + \psi(y, 0; \lambda) \int_{x \notin (\underline{a}, \bar{a})} \varphi(x, i(Z), Z)\mathrm{d}x, \quad Z \in \mathscr{B}. \tag{48}$$

We will further conjecture that:

$$\phi \in \mathscr{C}^1((\underline{a}, \underline{b}) \cup (\bar{b}, \bar{a})) \cap \mathscr{C}^0([\underline{a}, \bar{a}]).$$

This in turn provides the remaining conditions needed to solve the problem:

$$\lim_{x \nearrow y} \phi(y, y; \lambda) = \lim_{x \searrow y} \phi(y, y; \lambda), \quad \forall y \in [\underline{a}, \bar{a}] \tag{49}$$

$$\lim_{x \nearrow y} \phi'(y, y; \lambda) = \lim_{x \searrow y} \phi'(y, y; \lambda), \quad \forall y \in (\underline{a}, \underline{b}) \cup (\bar{b}, \bar{a}).$$

The idea is to solve the ODE in Eq. (46) piecewise over the relevant smoothness regions of Z and the integral equation in Eq. (47) over $\mathscr{B}$. For $y < \underline{b}$, there are three smoothness regions: $x \in (\underline{a}, y)$, $x \in (y, \underline{b})$, and $x \in (\bar{b}, \bar{a})$; Importantly, for $y \in \mathscr{B}$, the cdf is flat everywhere, except at 0. At $y = 0$ the density exhibits an atom, which we denote by A, because every time firms experiment

there is a finite probability that they end up exploring. Otherwise, i.e., $y \in \mathscr{B} \setminus \{0\}$, the density is flat as experimentation makes sure mass does not accumulate there; finally, for $y \in (\bar{b}, \bar{a})$, the smoothness regions are: $(\underline{a}, \underline{b})$, $(\bar{b}, y)$, and $(y, \bar{a})$. We thus solve the cdf for the cases $y < \underline{b}$ and $y > \bar{b}$, and then obtain the atom, A, by requiring that the cdf reaches 1. Starting with $y < \underline{b}$, let us index the regions $(\underline{a}, y)$, $(y, \underline{b})$, and $(\bar{b}, \bar{a})$ by (1), (2), and (3), respectively, and accordingly denote the solution to the ODE in Eq. (46) over these three regions by:

$$\psi^{(1)}(y, x; \lambda) = 1 + C_{1,l}(y, \lambda) e^{-x^2/2} H(-1 - 2\lambda/\tau_A, x/\sqrt{2}) + C_{2,l}(y, \lambda) e^{-x^2/2} M(-2\lambda/\tau_A, 1/2, -x^2/\sqrt{2}),$$

$$\psi^{(2)}(y, x; \lambda) = C_{1,m}(y, \lambda) e^{-x^2/2} H(-1 - 2\lambda/\tau_A, x/\sqrt{2}) + C_{2,m}(y, \lambda) e^{-x^2/2} M(-2\lambda/\tau_A, 1/2, -x^2/\sqrt{2}),$$

$$\psi^{(3)}(y, x; \lambda) = C_{1,h}(y, \lambda) e^{-x^2/2} H(-1 - 2\lambda/\tau_A, x/\sqrt{2}) + C_{2,h}(y, \lambda) e^{-x^2/2} M(-2\lambda/\tau_A, 1/2, -x^2/\sqrt{2}),$$

where $C_{1,l}$, $C_{2,l}$, $C_{1,m}$, $C_{2,m}$, $C_{1,h}$, and $C_{2,h}$ are six unknown integration coefficients. Furthermore, to write the solution to the integral equation in Eq. (48), we first define the density outside the experimentation region but inside the no-exploration region, i.e., within $\mathscr{A} \setminus \mathscr{B}$, as:

$$p(y, x; \lambda) \equiv \psi^{(1)}(y, x; \lambda) \mathbf{1}_{x \in (\underline{a}, y]} + \psi^{(2)}(y, x; \lambda) \mathbf{1}_{x \in (y, \underline{b})} + \psi^{(3)}(y, x; \lambda) \mathbf{1}_{x \in [\bar{b}, \bar{a})}.$$

Denoting the density over $\mathscr{B}$ by $\psi^{(4)}$ we can rewrite the integral equation in Eq. (48) as:

$$\psi^{(4)}(y, x; \lambda) = \int_{\mathscr{A} \setminus \mathscr{B}} p(y, x; \lambda) \varphi(x, i(Z), Z) \mathrm{d}x + \int_{\mathscr{B}} \psi^{(4)}(y, x; \lambda) \varphi(x, i(Z), Z) \mathrm{d}x$$
$$+ \psi^{(4)}(y, 0; \lambda) \int_{x \notin \mathscr{A}} \varphi(x, i(Z), Z) \mathrm{d}x$$

Iterating over this equation, define the following Liouville-Neumann series:

$$\theta(y, Z) \equiv \sum_{n=0}^{\infty} \theta^{(n)}(y, Z), \tag{50}$$

where $\theta^{(n)}$ satisfies for $n \in \mathbb{N}$:

$$\theta^{(n)}(y, Z) \equiv \int_{\mathscr{B}^n} \varphi(y, i(y^{(n)}), y^{(n)}) \varphi(y^{(n)}, i(y^{(n-1)}), y^{(n-1)}) \cdots \varphi(y^{(1)}, i(Z), Z) \mathrm{d}y^{(n)} \mathrm{d}y^{(n-1)} \cdots \mathrm{d}y^{(1)}.$$

Note that $\theta(y; Z)$ in Eq. (50) can be alternatively represented recursively as a Fredholm equation of the second kind:

$$\theta(y, Z) = \varphi(y, i(Z), Z) + \int_{\mathscr{B}} \theta(x, Z) \varphi(y, i(x), x) \mathrm{d}x.$$

The relevance of this alternative representation is that, given that φ is a probability density, the contraction mapping theorem (Theorem 3.2, Stokey and Lucas (1996)) guarantees that the representation as an infinite sum in Eq. (50) exists. We can then write the solution to Eq. (48) as:

$$\psi^{(4)}(y, Z; \lambda) = \int_{\mathscr{A} \backslash \mathscr{B}} p(y, x; \lambda)\theta(x, Z)\mathrm{d}x + \psi^{(4)}(y, 0; \lambda) \int_{x \notin \mathscr{A}} \theta(x, Z)\mathrm{d}x$$

Evaluating this expression at $x \equiv 0$ and solving for $\psi^{(4)}(y, 0; \lambda)$ we get an explicit solution of the form:

$$\psi^{(4)}(y, x; \lambda) = \int_{\mathscr{A} \backslash \mathscr{B}} p(y, x; \lambda)\theta(x, Z)\mathrm{d}x + \frac{\int_{\mathscr{A} \backslash \mathscr{B}} p(y, x; \lambda)\theta(x, 0)\mathrm{d}x}{1 - \int_{x \notin \mathscr{A}} \theta(x, 0)\mathrm{d}x} \int_{x \notin \mathscr{A}} \theta(x, Z)\mathrm{d}x. \qquad (51)$$

It will be subsequently convenient to simplify notations:

$$H(x; \lambda) \equiv e^{-x^2/2}H(-1 - 2\lambda/\tau_A, x/\sqrt{2}) \quad \text{and} \quad M(x; \lambda) \equiv e^{-x^2/2}M(-2\lambda/\tau_A, 1/2, -x^2/\sqrt{2}),$$

and define accordingly the following functions:

$$h_1(Z, \underline{x}, \overline{x}, \lambda) \equiv \int_{\underline{x}}^{\overline{x}} H(x; \lambda)\theta(x, Z)\mathrm{d}x \quad \text{and} \quad h_2(Z, \underline{x}, \overline{x}, \lambda) \equiv \int_{\underline{x}}^{\overline{x}} M(x; \lambda)\theta(x, Z)\mathrm{d}x,$$

the following probabilities:

$$\Theta(Z) \equiv \int_{x \notin \mathscr{A}} \theta(x, Z)\mathrm{d}x \quad \text{and} \quad \Phi(y, Z) \equiv \int_{\underline{a}}^{y} \theta(x, Z)\mathrm{d}x.$$

We then use these functions to further define:

$$\phi(y, Z) \equiv \Phi(y, Z) + \frac{\Theta(Z)}{1 - \Theta(0)}\Phi(y, 0)$$

$$g_k(Z, \underline{x}, \overline{x}, \lambda) \equiv h_k(Z, \underline{x}, \overline{x}, \lambda) + \frac{\Theta(Z)}{1 - \Theta(0)}h_k(0, \underline{x}, \overline{x}, \lambda), \quad k = 1, 2.$$

Using this simplified notation we can rewrite the solution in Eq. (51) as:

$$\psi^{(4)}(y, Z; \lambda) = \phi(y, Z) + \sum_{k=1,2} C_{l,k}(y, \lambda)g_k(Z, \underline{a}, y, \lambda)$$

$$+ \sum_{k=1,2} C_{m,k}(y, \lambda)g_k(Z, y, \underline{b}, \lambda) + \sum_{k=1,2} C_{h,k}(y, \lambda)g_k(Z, \overline{b}, \overline{a}, \lambda).$$

56

We finally obtain a piecewise representation of the density for $y < \underline{b}$:

$$\phi(y, x; \lambda) = \begin{cases} \psi^{(1)}(y, x; \lambda) & x \in (\underline{a}, y] \\ \psi^{(2)}(y, x; \lambda) & x \in (y, \underline{b}) \\ \psi^{(4)}(y, x; \lambda) & x \in \mathscr{B} \\ \psi^{(3)}(y, x; \lambda) & x \in (\bar{b}, \bar{a}) \end{cases},$$

Similarly, for $y > \bar{b}$, we can write:

$$\phi(y, x; \lambda) = \begin{cases} \psi^{(1)}(y, x; \lambda) & x \in (\underline{a}, \underline{b}) \\ \psi^{(4)}(y, x; \lambda) & x \in \mathscr{B} \\ 1 + \psi^{(2)}(y, x; \lambda) & x \in (\bar{b}, y] \\ \psi^{(3)}(y, x; \lambda) & x \in (y, \bar{a}) \end{cases},$$

where $\psi^{(4)}(y, x; \lambda)$ for $y > \bar{b}$ satisfies:

$$\psi^{(4)}(y, Z; \lambda) = \phi(y, Z) + \sum_{k=1,2} C_{l,k}(y, \lambda) g_k(Z, \underline{a}, \underline{b}, \lambda)$$

$$+ \sum_{k=1,2} C_{m,k}(y, \lambda) g_k(Z, \bar{b}, y, \lambda) + \sum_{k=1,2} C_{h,k}(y, \lambda) g_k(Z, y, \bar{a}, \lambda).$$

Given this piecewise representation of the cdf, Eqs. (47) and (49) constitute a system of 6 equations that pins down the 6 coefficients of integration, $C_{1,l}$, $C_{2,l}$, $C_{1,m}$, $C_{2,m}$, $C_{1,h}$, and $C_{2,h}$. The plan is to solve for these 6 coefficients and then recover the stationary CDF, π, through the Final Value Theorem:

$$\pi(y) \equiv \lim_{T \to \infty} \psi(y, x; T - t) = \lim_{\lambda \to 0} \lambda \phi(y, x; \lambda). \tag{52}$$

Imposing Eqs. (47) and (48) for $y \in (\underline{a}, \underline{b}) \cup (\bar{b}, \bar{a})$ yields:

$$\psi^{(1)}(y, \underline{a}; \lambda) = \psi^{(4)}(y, 0; \lambda), \tag{53}$$

$$\psi^{(3)}(y, \bar{a}; \lambda) = \psi^{(4)}(y, 0; \lambda),$$

$$\psi^{(4)}(y, \underline{b}; \lambda) = \psi^{(2)}(y, \underline{b}; \lambda)\mathbf{1}_{y<\underline{b}} + \psi^{(1)}(y, \underline{b}; \lambda)\mathbf{1}_{y>\bar{b}},$$

$$\psi^{(4)}(y, \bar{b}; \lambda) = \psi^{(3)}(y, \bar{b}; \lambda)\mathbf{1}_{y<\underline{b}} + (1 + \psi^{(2)}(y, \bar{b}; \lambda))\mathbf{1}_{y>\bar{b}},$$

$$\psi^{(2)}(y, y; \lambda) = \psi^{(1)}(y, y; \lambda)\mathbf{1}_{y<\underline{b}} + (\psi^{(3)}(y, y; \lambda) - 1)\mathbf{1}_{y>\bar{b}},$$

$$\psi^{(2)'}(y, y; \lambda) = \psi^{(1)'}(y, y; \lambda)\mathbf{1}_{y<\underline{b}} + \psi^{(3)'}(y, y; \lambda)\mathbf{1}_{y>\bar{b}}.$$

It is convenient to rewrite this system in matrix form, and then solve it separately for $y \in (\underline{a}, \underline{b})$ and $y \in (\bar{b}, \bar{a})$; we gather the 6 unknown coefficients in a vector and index them with a superscript

l for $y \in (\underline{a}, \underline{b})$ and h for $y \in (\overline{b}, \overline{a})$:

$$C_k(y; \lambda) \equiv \begin{pmatrix} C_{1l}^k(y, \lambda) & C_{2l}^k(y, \lambda) & C_{1m}^k(y, \lambda) & C_{2m}^k(y, \lambda) & C_{1h}^k(y, \lambda) & C_{2h}^k(y, \lambda) \end{pmatrix}', \quad k = \{l, h\}.$$

Using this notation, rewrite the system in Eq. (53) over the region $y \in (\underline{a}, \underline{b})$ as:

$$B_l(y) = A_l(y; \lambda)C_l(y; \lambda),$$

where

$$A_l(y; \lambda) = \begin{pmatrix}
\substack{H(\underline{a};\lambda) \\ -g_1(0,\underline{a},y,\lambda)} & \substack{M(\underline{a};\lambda) \\ -g_2(0,\underline{a},y,\lambda)} & -g_1(0, y, \underline{b}, \lambda) & -g_2(0, y, \underline{b}, \lambda) & -g_1(0, \overline{b}, \overline{a}, \lambda) & -g_2(0, \overline{b}, \overline{a}, \lambda) \\
-g_1(0, \underline{a}, y, \lambda) & -g_2(0, \underline{a}, y, \lambda) & -g_1(0, y, \underline{b}, \lambda) & -g_2(0, y, \underline{b}, \lambda) & \substack{H(\overline{a};\lambda) \\ -g_1(0,\overline{b},\overline{a},\lambda)} & \substack{M(\overline{a};\lambda) \\ -g_2(0,\overline{b},\overline{a},\lambda)} \\
g_1(\underline{b}, \underline{a}, y, \lambda) & g_2(\underline{b}, \underline{a}, y, \lambda) & \substack{g_1(\underline{b},y,\underline{b},\lambda) \\ -H(\underline{b};\lambda)} & \substack{g_2(\underline{b},y,\underline{b},\lambda) \\ -M(\underline{b});\lambda} & g_1(\underline{b}, \overline{b}, \overline{a}, \lambda) & g_2(\underline{b}, \overline{b}, \overline{a}, \lambda) \\
g_1(\overline{b}, \underline{a}, y, \lambda) & g_2(\overline{b}, \underline{a}, y, \lambda) & g_1(\overline{b}, y, \underline{b}, \lambda) & g_2(\overline{b}, y, \underline{b}, \lambda) & \substack{g_1(\overline{b},\overline{b},\overline{a},\lambda) \\ -H(\overline{b};\lambda)} & \substack{g_2(\overline{b},\overline{b},\overline{a},\lambda) \\ -H(\overline{b};\lambda)} \\
-H(y; \lambda) & -M(y; \lambda) & H(y; \lambda) & M(y; \lambda) & 0 & 0 \\
-H'(y; \lambda) & -M'(y; \lambda) & H'(y; \lambda) & M'(y; \lambda) & 0 & 0
\end{pmatrix},$$

and

$$B_l(y) = \begin{pmatrix} \phi(y, 0) - 1 & \phi(y, 0) & -\phi(y, \underline{b}) & -\phi(y, \overline{b}) & 1 & 0 \end{pmatrix}'.$$

Similarly, in the region $y \in (\overline{b}, \overline{a})$ we write:

$$B_h(y) = A_h(y; \lambda)C_h(y; \lambda),$$

where

$$A_h(y; \lambda) = \begin{pmatrix}
\substack{H(\underline{a};\lambda) \\ -g_1(0,\underline{a},\underline{b},\lambda)} & \substack{M(\underline{a};\lambda) \\ -g_2(0,\underline{a},\underline{b},\lambda)} & -g_1(0, \overline{b}, y, \lambda) & -g_2(0, \overline{b}, y, \lambda) & -g_1(0, y, \overline{a}, \lambda) & -g_2(0, y, \overline{a}, \lambda) \\
-g_1(0, \underline{a}, \underline{b}, \lambda) & -g_2(0, \underline{a}, \underline{b}, \lambda) & -g_1(0, \overline{b}, y, \lambda) & -g_2(0, \overline{b}, y, \lambda) & \substack{H(\overline{a};\lambda) \\ -g_1(0,y,\overline{a},\lambda)} & \substack{M(\overline{a};\lambda) \\ -g_2(0,y,\overline{a},\lambda)} \\
\substack{g_1(\underline{b},\underline{a},\underline{b},\lambda) \\ -H(\underline{b};\lambda)} & \substack{g_2(\underline{b},\underline{a},\underline{b},\lambda) \\ -M(\underline{b};\lambda)} & g_1(\underline{b}, \overline{b}, y, \lambda) & g_2(\underline{b}, \overline{b}, y, \lambda) & g_1(\underline{b}, y, \overline{a}, \lambda) & g_2(\underline{b}, y, \overline{a}, \lambda) \\
g_1(\overline{b}, \underline{a}, \underline{b}, \lambda) & g_2(\overline{b}, \underline{a}, \underline{b}, \lambda) & \substack{g_1(\overline{b},\overline{b},y,\lambda) \\ -H(\overline{b};\lambda)} & \substack{g_2(\overline{b},\overline{b},y,\lambda) \\ -M(\overline{b};\lambda)} & g_1(\overline{b}, y, \overline{a}, \lambda) & g_2(\overline{b}, y, \overline{a}, \lambda) \\
0 & 0 & H(y; \lambda) & M(y; \lambda) & -H(y; \lambda) & -M(y; \lambda) \\
0 & 0 & H'(y; \lambda) & M'(y; \lambda) & -H'(y; \lambda) & -M'(y; \lambda)
\end{pmatrix},$$

and

$$B_h(y) = \begin{pmatrix} \phi(y, 0) - 1 & \phi(y, 0) & 1 - \phi(y, \underline{b}) & 1 - \phi(y, \overline{b}) & -1 & 0 \end{pmatrix}'.$$

We can then write the solution in the two regions as:

$$C_k(y, \lambda) = A_k(y; \lambda)^{-1} B_k(y), \quad k = \{l, h\}.$$

Since we are interested in the stationary distribution only, the last step is to apply the Final Value Theorem. This implies taking the limit in Eq. (52). Note that

$$\lim_{\lambda \to 0} H(x; \lambda) = \frac{\sqrt{\pi}}{2} (1 - \text{erf}(x/\sqrt{2})),$$
$$\lim_{\lambda \to 0} M(y; \lambda) = 1,$$

and that at a stationary state we lose the dependence on the initial state, which implies that (and which can be verified by direct computation):

$$\lim_{\lambda \to 0} C_{1,l}(y, \lambda) \equiv \lim_{\lambda \to 0} C_{1,m}(y, \lambda) \equiv \lim_{\lambda \to 0} C_{1,h}(y, \lambda) \equiv 0,$$
$$\lim_{\lambda \to 0} C_{2,l}(y, \lambda) \equiv \lim_{\lambda \to 0} C_{2,m}(y, \lambda) \equiv \lim_{\lambda \to 0} C_{2,h}(y, \lambda).$$

These observations, along with Eq. (52), imply that we can write the stationary cdf as:

$$\pi(y) \equiv \lim_{\lambda \to 0} 1 + C_{2,l}(y, \lambda) \equiv \lim_{\lambda \to 0} C_{2,m}(y, \lambda) \equiv \lim_{\lambda \to 0} C_{2,h}(y, \lambda).$$

To compute these limits we rewrite the solution for coefficients C as:

$$C_k(y, \lambda) = \frac{1}{\det(A_k(y;\lambda))} \text{adj}(A_k(y; \lambda)) B_k(y), \quad k = \{l, h\},$$

where $\text{adj}(A)$ denotes the adjugate of A. Note that the dependence on λ is both in the denominator through the determinant, and in the numerator through the adjugate matrix; accordingly we apply L'Hospital Theorem to write this limit as:

$$\lim_{\lambda \to 0} C_k(y, \lambda) = \frac{\lim_{\lambda \to 0} \frac{d}{d\lambda} \text{adj}(A_k(y;\lambda)) B_k(y)}{\lim_{\lambda \to 0} \frac{d}{d\lambda} \det(A_k(y;\lambda))}, \quad k = \{l, h\}.$$

E   Distribution of the Length of the Exploration Cycle

We derive the distribution of the length of the exploration cycle based on a system of coupled Kolmogorov Backward Equations. Formally, our goal is to compute the probability that the first exploration time θ (defined in Eq. (9)) occurs before or at time $t \geq 0$ given that Z_0 starts at 0 (i.e., immediately following an initial round of exploration):

$$G(t) \equiv \mathbb{P}(\theta \leq t | Z_0 = 0).$$

To keep notations transparent, for this appendix only we change notations for stopping times and define the hitting time of Z at level k as:

$$\tau_k \equiv \inf\{t : Z_t = k\}.$$

To derive an expression for $G(\cdot)$, it is convenient to start by working piecewise over two regions, $(\underline{a}, \underline{b})$ and $(\bar{b}, \bar{a})$; accordingly, define the two functions:

$$f_1(t, Z) \equiv \mathbb{P}\left(\theta \leq t | Z_0 \in (\bar{b}, \bar{a})\right) = \mathbb{E}\left(\mathbf{1}_{\theta \leq t} | Z_0 \in (\bar{b}, \bar{a})\right),$$
$$f_2(t, Z) \equiv \mathbb{P}\left(\theta \leq t | Z_0 \in (\bar{b}, \bar{a})\right) = \mathbb{E}\left(\mathbf{1}_{\theta \leq t} | Z_0 \in (\bar{b}, \bar{a})\right).$$

These two functions satisfy a system of coupled equations. Suppose first that today Z_0 falls into the first region, $(\underline{a}, \underline{b})$. In this situation, there are two ways exploration occurs. Either we hit $\underline{a}$ first, or we hit $\underline{b}$ first and with probability:

$$\underline{\Theta} \equiv \int_{y \notin \mathscr{A}} \theta(y, \underline{b}) \mathrm{d}y,$$

experimentation leads us to explore. Note that $\underline{\Theta}$ takes into account all possible number of successive experimentation rounds that leads us to explore; it is thus based on the function θ defined in Eq. (50). Furthermore, upon hitting $\underline{b}$ first, multiple successive rounds of experimentation may either take us back to the first region $(\underline{a}, \underline{b})$ or to the second region $(\bar{b}, \bar{a})$, which precisely couples the system of equations for $f_1(\cdot)$ and $f_2(\cdot)$. Similarly, when starting in the second region $Z_0 \in (\bar{b}, \bar{a})$, exploration may either occur as we hit $\bar{a}$ first or as we hit $\bar{b}$ first and then with probability:

$$\bar{\Theta} \equiv \int_{y \notin \mathscr{A}} \theta(y, \bar{b}) \mathrm{d}y,$$

successive rounds of experimentation lead to exploration. Formally, they satisfy:

$$f_1(t, Z) = \mathbb{E}\left[\mathbf{1}_{\tau_{\underline{a}} \leq t} \mathbf{1}_{\tau_{\underline{a}} < \tau_{\underline{b}}} | Z \in (\underline{a}, \underline{b})\right] + \mathbb{E}\left[\mathbf{1}_{\tau_{\underline{b}} < \tau_{\underline{a}}} \left(\begin{array}{c} \underline{\Theta}\mathbf{1}_{\tau_{\underline{b}} \leq t} + \int_{\underline{a}}^{\underline{b}} f_1(t, x)\theta(x, \underline{b})\mathrm{d}x \\ + \int_{\bar{b}}^{\bar{a}} f_2(t, x)\theta(x, \underline{b})\mathrm{d}x \end{array} \right) \middle| Z \in (\underline{a}, \underline{b}) \right] \quad (54)$$

$$f_2(t, Z) = \mathbb{E}\left[\mathbf{1}_{\tau_{\bar{a}} \leq t} \mathbf{1}_{\tau_{\bar{a}} < \tau_{\bar{b}}} | Z \in (\bar{b}, \bar{a})\right] + \mathbb{E}\left[\mathbf{1}_{\tau_{\bar{b}} < \tau_{\bar{a}}} \left(\begin{array}{c} \bar{\Theta}\mathbf{1}_{\tau_{\bar{b}} \leq t} + \int_{\underline{a}}^{\underline{b}} f_1(t, x)\theta(x, \bar{b})\mathrm{d}x \\ + \int_{\bar{b}}^{\bar{a}} f_2(t, x)\theta(x, \bar{b})\mathrm{d}x \end{array} \right) \middle| Z \in (\bar{b}, \bar{a}) \right].$$

We further define the following functions:

$$\theta_1(Z) \equiv \mathbb{E}\left[\mathbf{1}_{\tau_{\underline{b}} < \tau_{\underline{a}}} \middle| Z \in (\underline{a}, \underline{b})\right],$$
$$\theta_2(Z) \equiv \mathbb{E}\left[\mathbf{1}_{\tau_{\bar{b}} < \tau_{\bar{a}}} \middle| Z \in (\bar{b}, \bar{a})\right].$$

These expressions satisfy the following ODE over their respective range:

$$0 = \theta_{k,Z} + \theta_{k,ZZ}, \quad k = 1, 2,$$

with boundary conditions $\theta_1(\underline{b}) = \theta_2(\overline{b}) = 1$ and $\theta_1(\underline{a}) = \theta_2(\overline{a}) = 0$, respectively. The solution is:

$$\theta_1(Z) = \frac{e^{\underline{b}}\left(e^{\underline{a}-Z}-1\right)}{e^{\underline{a}}-e^{\underline{b}}} \quad \text{and} \quad \theta_2(Z) = \frac{e^{\overline{b}}\left(e^{\overline{a}-Z}-1\right)}{e^{\overline{a}}-e^{\overline{b}}} \ .$$

Similarly, we define the functions:

$$\phi_1(t,Z) \equiv \mathbb{E}\left[\mathbf{1}_{\tau_{\underline{a}}\leq t}\mathbf{1}_{\tau_{\underline{a}}<\tau_{\underline{b}}}\middle| Z \in (\underline{a},\underline{b})\right], \quad \phi_2(t,Z) \equiv \mathbb{E}\left[\mathbf{1}_{\tau_{\overline{a}}\leq t}\mathbf{1}_{\tau_{\overline{a}}<\tau_{\overline{b}}}\middle| Z \in (\overline{b},\overline{a})\right],$$
$$\psi_1(t,Z) \equiv \mathbb{E}\left[\mathbf{1}_{\tau_{\underline{b}}\leq t}\mathbf{1}_{\tau_{\tau_{\underline{b}}}<\underline{a}}\middle| Z \in (\underline{a},\underline{b})\right] \quad \psi_2(t,Z) \equiv \mathbb{E}\left[\mathbf{1}_{\tau_{\overline{b}}\leq t}\mathbf{1}_{\tau_{\tau_{\overline{b}}}<\overline{a}}\middle| Z \in (\overline{b},\overline{a})\right].$$

These expressions satisfy the following PDE over their respective range:

$$\phi_{k,t} = \tau_A/2(\phi_{k,Z} + \phi_{k,ZZ}) \quad \phi_k(0,Z) = 0, \quad k = 1,2,$$
$$\psi_{k,t} = \tau_A/2(\psi_{k,Z} + \phi_{k,ZZ}) \quad \psi_k(0,Z) = 0, \quad k = 1,2,$$

with boundary conditions:

$$\phi_1(t,\underline{a}) = 1, \quad \phi_1(t,\underline{b}) = 0,$$
$$\phi_2(t,\overline{a}) = 1, \quad \phi_2(t,\overline{b}) = 0,$$
$$\psi_1(t,\underline{b}) = 1, \quad \psi_1(t,\underline{a}) = 0$$
$$\psi_2(t,\overline{b}) = 1, \quad \psi_2(t,\overline{a}) = 0.$$

Although these expressions do not have analytical solutions, their Laplace transforms do. Denoting with a $\widehat{g}$ the Laplace transform of a function g and denoting by λ its frequency, we have:

$$\widehat{\phi}_1(\lambda,Z) = \left(e^{\frac{(\underline{a}-Z)(\sqrt{8\lambda+\tau_A}+\sqrt{\tau_A})}{2\sqrt{\tau_A}}}\left(e^{\frac{Z\sqrt{8\lambda+\tau_A}}{\sqrt{\tau_A}}} - e^{\frac{\underline{b}\sqrt{8\lambda+\tau_A}}{\sqrt{\tau_A}}}\right)\right)\bigg/\left(e^{\frac{\underline{a}\sqrt{8\lambda+\tau_A}}{\sqrt{\tau_A}}} - e^{\frac{\underline{b}\sqrt{8\lambda_A+\tau}}{\sqrt{\tau_A}}}\right),$$

$$\widehat{\phi}_2(\lambda,Z) = \left(e^{\frac{(\overline{a}-Z)(\sqrt{8\lambda+\tau_A}+\sqrt{\tau_A})}{2\sqrt{\tau_A}}}\left(e^{\frac{Z\sqrt{8\lambda+\tau_A}}{\sqrt{\tau_A}}} - e^{\frac{\overline{b}\sqrt{8\lambda+\tau_A}}{\sqrt{\tau_A}}}\right)\right)\bigg/\left(e^{\frac{\overline{a}\sqrt{8\lambda+\tau_A}}{\sqrt{\tau_A}}} - e^{\frac{\overline{b}\sqrt{8\lambda_A+\tau}}{\sqrt{\tau_A}}}\right),$$

$$\psi_1(\lambda,Z) = \left(\left(e^{\frac{\underline{a}\sqrt{8\lambda+\tau_A}}{\sqrt{\tau_A}}} - e^{\frac{Z\sqrt{8\lambda+\tau_A}}{\sqrt{\tau_A}}}\right)e^{\frac{(\underline{b}-Z)(\sqrt{8\lambda+\tau_A}+\sqrt{\tau_A})}{2\sqrt{\tau_A}}}\right)\bigg/\left(e^{\frac{\underline{a}\sqrt{8\lambda+\tau_A}}{\sqrt{\tau_A}}} - e^{\frac{\underline{b}\sqrt{8\lambda+\tau_A}}{\sqrt{\tau_A}}}\right),$$

$$\psi_2(\lambda,Z) = \left(\left(e^{\frac{\overline{a}\sqrt{8\lambda+\tau_A}}{\sqrt{\tau_A}}} - e^{\frac{Z\sqrt{8\lambda+\tau_A}}{\sqrt{\tau_A}}}\right)e^{\frac{(\overline{b}-Z)(\sqrt{8\lambda+\tau_A}+\sqrt{\tau_A})}{2\sqrt{\tau_A}}}\right)\bigg/\left(e^{\frac{\overline{a}\sqrt{8\lambda+\tau_A}}{\sqrt{\tau_A}}} - e^{\frac{\overline{b}\sqrt{8\lambda+\tau_A}}{\sqrt{\tau_A}}}\right).$$

We can then recover each function through Laplace inversion. We finally rewrite the system in Eq. as:

$$f_1(t,Z) = \phi_1(t,Z) + \underline{\Theta}\psi_1(t,Z) + \theta_1(Z)\left(\int_{\underline{a}}^{\underline{b}} f_1(t,x)\theta(x,\underline{b})\mathrm{d}x + \int_{\overline{b}}^{\overline{a}} f_2(t,x)\theta(x,\underline{b})\mathrm{d}x\right)$$

$$f_2(t,Z) = \phi_2(t,Z) + \overline{\Theta}\psi_2(t,Z) + \theta_2(Z)\left(\int_{\underline{a}}^{\underline{b}} f_1(t,x)\theta(x,\overline{b})\mathrm{d}x + \int_{\overline{b}}^{\overline{a}} f_2(t,x)\theta(x,\overline{b})\mathrm{d}x\right).$$

Multiply first both sides of each equation by $\theta(x, \underline{b})$, integrate the first one over the first region, the second one over the second region, add these two equations together; then multiply both sides of each equation by $\theta(x, \overline{b})$ and repeat the same operations. The resulting system of equations yields:

$$
\begin{pmatrix}
\int_{\underline{a}}^{\underline{b}} f_1(t,x)\theta(x,\underline{b})\mathrm{d}x + \int_{\overline{b}}^{\overline{a}} f_2(t,x)\theta(x,\underline{b})\mathrm{d}x \\
\int_{\underline{a}}^{\underline{b}} f_1(t,x)\theta(x,\overline{b})\mathrm{d}x + \int_{\overline{b}}^{\overline{a}} f_2(t,x)\theta(x,\overline{b})\mathrm{d}x
\end{pmatrix}
= -
\begin{pmatrix}
\int_{\underline{a}}^{\underline{b}} \theta_1(x)\theta(x,\underline{b})\mathrm{d}x - 1 & \int_{\overline{b}}^{\overline{a}} \theta_2(x)\theta(x,\underline{b})\mathrm{d}x \\
\int_{\underline{a}}^{\underline{b}} \theta_1(x)\theta(x,\overline{b})\mathrm{d}x & \int_{\overline{b}}^{\overline{a}} \theta_2(x)\theta(x,\overline{b})\mathrm{d}x - 1
\end{pmatrix}^{-1}
$$

$$
\begin{pmatrix}
\int_{\underline{a}}^{\underline{b}} (\phi_1(t,x) + \underline{\Theta}\psi_1(t,x))\theta(x;\underline{b})\mathrm{d}s + \int_{\overline{b}}^{\overline{a}} (\phi_2(t,x) + \overline{\Theta}\psi_2(t,x))\theta(x;\underline{b})\mathrm{d}s \\
\int_{\underline{a}}^{\underline{b}} (\phi_1(t,x) + \underline{\Theta}\psi_1(t,x))\theta(x;\overline{b})\mathrm{d}s + \int_{\overline{b}}^{\overline{a}} (\phi_2(t,x) + \overline{\Theta}\psi_2(t,x))\theta(x;\overline{b})\mathrm{d}s
\end{pmatrix}.
$$

Substituting back into Eq. (54) delivers an explicit solution for the two functions $f_1(\cdot)$ and $f_2(\cdot)$. Finally, using these functions we compute the cdf when starting in the experiment region, $Z \in \mathscr{B}$:

$$
f_3(t, Z) \equiv \int_{\underline{a}}^{b} f_1(t,x)\varphi(x, i(Z), Z)\mathrm{d}x + \int_{\overline{b}}^{\overline{a}} f_2(t,x)\varphi(x, i(Z), Z)\mathrm{d}x
$$

$$
+ \int_{\mathscr{B}} f_3(t,x)\varphi(x, i(Z), Z)\mathrm{d}x + \int_{x \notin \mathscr{A}} \varphi(x, i(Z), Z)\mathrm{d}x.
$$

Iterating over this relation and evaluating it at $Z = 0$ we obtain the desired cdf:

$$
G(t) \equiv f_3(t, 0) = \int_{\underline{a}}^{b} f_1(t,x)\theta(x, 0)\mathrm{d}x + \int_{\overline{b}}^{\overline{a}} f_2(t,x)\theta(x, 0)\mathrm{d}x + \int_{x \notin \mathscr{A}} \theta(x, 0)\mathrm{d}x.
$$